Some Applications of Advanced Nonlinear Control Techniques

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Abstract

With the development of nonlinear control theories, the idea to apply nonlinear control techniques to physical systems has tremendous possibilities and more advantages than linear control methods. However, because of the complexity and limitation of nonlinear control theories, it is not straightforward to use them in most practical engineering systems.

The aim of this thesis is to apply nonlinear output regulation theory and nonlinear H_∞ control laws on practical engineering systems and evaluate the results with numerical simulation. The thesis consists of three parts.

In the first part, we consider the attitude control problem of spacecraft. The problem is complex since the mass of the spacecraft may be uncertain and external disturbances exist. We will apply the general framework of output regulation to solve this nonlinear control problem.

Here, the attitude of spacecraft is represented by the unit quaternion instead of Euler angles, or other three-parameter representations of orientation angles because the three-parameter representations always exhibit singularities at some spatial orientations. We formulate the unit quaternion-based control problem with output regulation theory. The desired angular velocity and disturbance are supposed to be produced by some autonomous systems.

We have obtained a design by overcoming three major obstacles: find a closed-form solution of the regulator equations and show that the solution of the regulator equations satisfies certain required conditions, devise an appropriate nonlinear internal model and form the augmented system composed of the given plant and the internal model, and globally stabilize the augmented system. Two cases are studied, respectively. The first case assumes that the system is exactly known and the second case takes into account parameter variations of the spacecraft model. Controllers are designed to achieve global asymptotic attitude tracking of the desired attitude motions while the spacecraft is subject to an external disturbance. The design results are evaluated through computer simulations.

In the second part, we consider the disturbance attenuation problem of aircraft control system in severe windshears by approximate continuous-time H_∞ control law.

The nonlinear H_∞ control has been studied by lots of scholars and the problem comes down to the solution of Hamilton-Jacobi-Isaacs (HJI) equation. Because of the nonlinear nature of equation, it is usually impossible to obtain a closed form solution of HJI. In this part, we will apply an approximation control law to the practical system to achieve disturbance attenuation.

The second-order and third-order approximations of the continuous-times nonlinear H_∞ control for aircraft control system in windshears are designed. The control laws are evaluated and

compared with the linear H_∞ control law by computer simulation.

In the third part, we consider the disturbance attenuation of the nonlinear benchmark system by approximate discrete-time nonlinear H_∞ control. Like the continuous-time case, the discrete-time nonlinear H_∞ problem comes down to the solution of the discrete Hamilton-Jacobi-Isaacs (DHJI) equation. Nowadays the approximation control law is the only feasible way in practical design because of the same reason raised in the continuous-time case.

In this part, a third-order approximation of the discrete-time nonlinear H_∞ control is obtained and compared with the linear discrete-time H_∞ control law by computer simulation. The third-order control law has shown a better performance than the linear one.

摘要

随着非线性控制理论的发展，试图把非线性控制技术运用到实际工程系统中的想法变得越来越具有可行性，而且与传统的线性控制方法比较，非线性控制具有更多的优势。然而，非线性控制理论的复杂性和局限性加深了这个论题的难度。

本论文旨在运用非线性输出调节理论和非线性 H 无穷控制率去解决一些实际的工程问题，并且用数字仿真来评估设计结果。本文由三部分组成。

第一部分探讨空间飞行器的姿态控制问题。飞行器质量的不确定性，以及外部干扰的存在，使得这个问题具有了一定的难度。我们将运用输出调节的一般性解决方法来处理这个非线性控制问题。在本文中，空间飞行器的姿态用单位四元数来表示，而不是用欧拉角或者其他三参数方法，原因在于用三参数方法表示总是会在某些特定方位出现奇点。这里我们用输出调节的理论来描述这个以单位四元法表示的控制问题，并且假设期望的角速度和干扰是由某些自治系统产生的。在设计控制器的过程中我们主要克服了三个方面的困难。首先，找到调节器方程的闭型解，并说明这个解满足特定的条件；然后，设计一个合适的非线性内模，给定的系统与这个内模一起组成一个增广系统；最后，全局镇定增广系统。这里分别考虑了两种情况：第一种情况假定系统是完全已知的，第二种情况考虑到了空间飞行器模型的参数不确定性。设计的控制器可以完成在有外界干扰情况下的空间飞行器的全局渐进姿态跟踪，跟踪的目标是期望的姿态运动。设计的结果由计算机仿真来评估。

在第二部分中，我们考虑了风切变情况下的飞行器控制系统的干扰抑制问题。这里我们运用了近似的连续 H 无穷控制律。许多学者已经深入地探讨了非线性 H 无穷控制问题，并且把这个问题归结为求一个 HJI 方程的解。由于该方程的非线性特性，一般情况下我们不可能得到其闭型解。所以，我们将会运用到一种近似的控制方法去达到干扰抑制的目的。我们设计出在风切变情况下飞行器控制系统的连续非线性 H 无穷控制率的二次和三次近似解。最后，我们用计算机仿真来并评估所设计的近似非线性控制率，并且用其与线性控制率进行比较。

在第三部分中，通过运用近似的离散非线性 H 无穷控制方法，我们解决了非线性基准系统的干扰抑制问题。和连续系统一样，离散的非线性 H 无穷控制问题的解归结为求一个 DHJI 方程的解的问题。由于和连续系统相同的原因，求近似控制率是目前实际设计中唯一可行的方法。这里我们得到了一个离散非线性 H 无穷控制器的三次近似解，并且通过计算机仿真将它与线性控制器进行比较。在离散的非线性基准系统中，三次近似控制率表现要好于线性控制率。

Contents

Chapter 1

Introduction

Nowadays linear control techniques have been widely used in practical engineering systems, thanks to the maturity of development of linear control theory and the simplicity of applications. However, with the improvement of technology and development of society, the requirements on engineering objectives are much stricter, which usually are highly nonlinear and coupled. Even though the theories of linear systems have provided powerful design and analysis methods, the behaviors of the nonlinear engineering systems may be more complex. Neglecting nonlinear parts of engineering systems are common technique for linear controller designs, but it also brings unpredictable stability and bad performances. Some simple examples in Chapter 1 of [2] highlight the shortcomings of applying linear control techniques on nonlinear systems, including limitation of linearization, loss of tracking performance and peaking phenomena.

With the development of nonlinear control theory, more and more nonlinear control techniques are used in practical engineering systems. Compared to linear controllers, nonlinear controllers can achieve better stability, better system behaviors and save energy. Moreover nonlinear control techniques can stabilize or tracking some nonlinear systems, but linear control techniques fail. However, the pitfalls of nonlinear control techniques are also obvious. The nonlinear controller design is not as systematic as the linear one. For different systems, different nonlinear theories may be applied. Due to the complexity of nonlinear theories, the design methods are usually difficult and conditionally limited. The success of nonlinear controller design often depends on the satisfaction of strict solvable conditions. That is to say, even though nonlinear control theories are self-contained, the applications on physical systems are still challenging.

Motivated by the desire to overcome this kind of challenges, in this thesis, I would like to apply nonlinear control techniques on some physical systems. The nonlinear control methods which will be used here are separately the general framework of output regulation theorem, approximation continuous-time nonlinear H_∞ control law, and approximation discrete-time nonlinear H_∞ control law. This chapter provides the overviews of theoretical backgrounds of these techniques and introduces the object engineering systems.

1.1 Overview of Output Regulation Problem

The *output regulation problem* means to design a feedback controller to achieve asymptotic tracking for a class of reference inputs and reject a class of disturbance in an uncertain system while keeping the closed-loop system stable. Here a classic control problem shown in Figure 1.1. gives a close look at the description of output regulation problem. A controller $u(t)$ is designed so that the closed-loop system is exponentially stable and the output $y(t)$ of the plant can asymptotically track a reference input $r(t)$ in the presence of the disturbance $d(t)$. The problem can be called asymptotic tracking and disturbance rejection problem. When the reference inputs and disturbance are assumed to be generated by some autonomous differential equations called exosystem, the problem is named as output regulation problem or servomechanism problem.

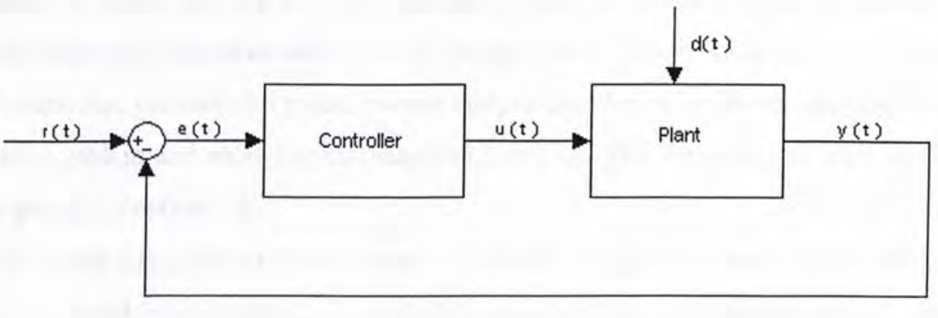

Figure 1.1: Output Regulation Problem.

When we deal with the output regulation problem for uncertain nonlinear systems with time-varying reference inputs and disturbance, three important issues [8] need to be paid attention on. First, "how to define and guarantee existence of the steady state of the system, and hence characterize the solvability of the problem"; Second, "how to handle the plant uncertainties"; And third, how to get the *global output regulation* result, that means to achieve asymptotic tracking and disturbance rejection in a nonlinear system with uncertain parameters arbitrarily large but bounded, where the initial states of the plant, the exosystem and the controller are arbitrarily large. These three questions are entirely different from possible situations in the linear output regulation problems and we could not find the clues directly from the linear cases.

The idea to solve output regulation problems actually is to convert it to a stabilization problem. To tell a nonlinear output regulation problem is solvable or not, first, we need to consider the solvability of *Regulator Equations*. The solution of regulator equations in fact characterizes the steady states of the system. If only we could find the steady states of the system, it is possible to convert the output regulation problem into stabilization problem.

Linear internal model principle plays an important role on the linear robust output regulation problem due to the presence of the uncertain parameter and static state feedback control law useless. Though it fails to work in general nonlinear cases because it fails to handle the nonlinear system whose regulator equations have non-polynomial solutions and it is difficult to get global stability, it motivates the appearance of nonlinear internal model. Choosing an appropriate nonlinear internal model can solve the second problem ahead.

For the third problem, research is case by case because of the complexity of nonlinear systems. In fact, global robust stabilization problem itself is an open topic.

Considering these three problems, the general framework to solve global robust output regulation problem, provided by Huang and Chen in [7], includes three steps: First, solve the regulator equations; Second, find an appropriate steady-state generator for the system and design a generalized internal model based on the generator; Third, after some coordinate and input transformations, convert the global robust output regulation problem into a global robust stabilization problem and solve the stabilization problem. The detailed summary of the framework will be given in Section 2.3

In [7], it was also presented under what conditions the global robust output regulation problem can be converted into a global robust stabilization problem for an augmented system. On the other hand, the framework relaxes the assumption of the solution of the regulator equations and gives more flexibility to use global stabilization methods. However, there are also challenges for using this method, because the global robust stabilization problem itself is a challenging topic, the regulator equations are sometimes not solvable, the assumption of the solutions of regulator equations is limited and the process to find an appropriate nonlinear internal model is complex. In this thesis we will apply the framework to the attitude tracking control problem of rigid spacecraft.

1.2 Attitude Tracking Control of Rigid Spacecraft

With the development of space technology, spacecrafts are used widely. The need of high degree of accuracy and the existence of uncertainty of systems bring about the spacecraft control problems difficulties. As a complex system, there are lots of topics in spacecraft control, such as spacecraft attitude control, control for spacecraft formation flying, stabilization of unstable spacecraft dynamics, guidance design and so on. In this thesis we mainly focus on spacecraft attitude control problem.

Many present and future space missions are required to track a time-varying reference trajectory rather than a simple set point regulation. The equations of the systems are nonlinear and coupled. The complexity of the problem is exacerbated by the fact that the mass properties of the spacecraft may be uncertain due to payload variation, rotation of solar arrays or fuel consumption, and the fact that external disturbances need to be considered, which are caused by space environment and include mainly gravity-gradient torque, aerodynamic torque, solar radiation torque and magnetic torque.

In [6] and [11], the tracking problem is solved via PD-like control laws, where no disturbance and no uncertainty are considered. If uncertainties exist, adaptive control methods [10] are widely used. In [3], a passivity-based adaptive control method was designed to achieve attitude tracking with a global convergence. In [1] an adaptive feedback control laws for zero-disturbance spacecraft was presented to guarantee global asymptotic attitude tracking problem based on control Lyapunov function. In [2], adaptive full-state feedback controller and adaptive output feedback controller are developed respectively to the quaternion-based attitude tracking control problem. More recently, Luo and Chu provided an inverse optimal adaptive control for attitude tracking with a global convergence using integrator backstepping and the disturbance attenuation in [5] and [4]. Though adaptive methods could handle the parameter uncertainty, the unbounded energy disturbance asymptotic rejection was still unsolved in these papers.

Since it is known that output tracking and disturbance rejection are exactly the objectives of the output regulation problem, we will formulate the control problem with output regulation theory and use the general framework mentioned above to solve it.

1.3 Overview of Continuous-time Nonlinear H_∞ Control

Another nonlinear control technique we will apply is the continuous-time nonlinear H_∞ control. There have been considerable studies in developing nonlinear H_∞ control theory in [15][16][17] and [18]. In these references, the disturbance attenuation problem was provided and solved under the assumption that there exists a positive-definite solution to the so called Hamilton-Jacobi-Isaacs equation. The disturbance attenuation problem means to design the feedback control input so that the closed-loop system asymptotically stable at the origin and the influence of exogenous input, such as disturbances and reference commands, on the penalty variable is attenuated. In the following we will describe the state feedback nonlinear H_∞ control problem [16][17].

Consider the system,

$$\begin{aligned} \dot{x} &= f(x) + g_1(x)w + g_2(x)u \\ z &= h_1(x) + k_{12}(x)u \end{aligned} \tag{1.1}$$

where $x \in R^n$ is the plant state, $u \in R^{m_2}$ is the plant input, $w \in R^{m_1}$ is a set of exogenous input variables, $z \in R^p$ is a penalty variable, and all functions are smooth, and defined in a neighborhood of the origin. Without loss of generality, we assume $f(0) = 0$ and $h_1(0) = 0$.

In order to simplify the calculation, we also assume

$$\begin{aligned} h_1^T(x)k_{12}(x) &= 0 \\ k_{12}^T(x)k_{12}(x) &= R_2 \end{aligned} \tag{1.2}$$

with R_2 a nonsingular constant matrix.

The objective of the H_∞ control is to design a controller with the form:

$$u = k(x), \tag{1.3}$$

where $k(x)$ is a locally defined sufficiently smooth function satisfying $k(0) = 0$, so that 1) when $w(t) = 0$, the closed-loop system is asymptotically stable at the origin and 2) given a real number $\gamma > 0$, for zero initial state $x(0) = 0$, for every $T > 0$ and for every piecewise continuous function $w : [0, T] \to R^{m_1}$

$$\int_0^T z^T(s)z(s)ds \leq \gamma^2 \int_0^T w^T(s)w(s)ds, \tag{1.4}$$

where, this condition is also defined as disturbance attenuation.

The control problem was theoretically solved in [16] to [18] and we will summary the solution here:

If there exists a positive definite C^1 function $V(x)$, locally defined in a neighborhood of the origin in R^n that satisfies the following Hamilton-Jacobi-Isaacs (HJI) equation:

$$V_x f(x) + \frac{1}{2}h_1^T h_1 + \frac{1}{2}V_x\left(\frac{g_1 g_1^T}{\gamma^2} - g_2 R_2^{-1} g_2\right)V_x^T = 0 \tag{1.5}$$

where V_x denotes the Jacobian matrix of $V(x)$, then the state feedback control law given by

$$u = -R_2^{-1}(x)g_2^T(x)V_x^T \tag{1.6}$$

solves the disturbance attenuation problem with performance level specified by γ.

The main block to apply this control law to physical systems is that it is usually impossible to find a closed form solution for HJI equation. So we bypass this difficulty and solve the problem with the approximation method. An approximation approach to solve HJI equation in terms of the Taylor series was proposed in [24] and we will summarize it in Section 3.1.

In this thesis, we will apply the approximation algorithm to design an approximate continuous-time nonlinear H_∞ control law to achieve disturbance attenuation for flight control system in windshears.

1.4 Overview of Discrete-time Nonlinear H_∞ Control

The last nonlinear control technique we will use is the discrete-time H_∞ control law. Like the continuous-time case, the discrete-time H_∞ control problem has been well studied in several papers [19], [22], [28]. Here we describe the control problem as follows.

Consider the following discrete-time nonlinear systems

$$\begin{aligned} x(t+1) &= A(x(t)) + B(x(t))u(t) + E(x(t))w(t), \\ & \quad t = 0, 1, 2, \cdots \\ z(t) &= C(x(t)) + D(x(t))u(t) + F(x(t))w(t) \end{aligned} \tag{1.7}$$

where $x \in R^n$ is the plant state, $u \in R^m$ is the plant input, $w \in R^r$ is an exogenous input, and $z \in R^p$ is a penalty variable. It is assumed that all functions involved in this setup are smooth and defined in a neighborhood U of the origin in R^n. Without loss of generality, we assume $A(0) = 0$ and $C(0) = 0$.

We hope to design a feedback control law of the form

$$u = k(x, w) \tag{1.8}$$

such that the closed-loop system is 1) locally internal stable, i.e., when $w(t) = 0$, the closed-loop discrete-time system is asymptotically stable at the origin and 2) the L_2 gain of the disturbance input w to the penalty variable z is attenuated by a prescribed real number γ. Here we need to notice that the controller (1.8) is somehow depending on the disturbance input, which is different from the continuous-time case.

The control problem has been theoretically solved in [19], [22], [28]. We summary it here:

If there exists a smooth positive definite function $V(x)$, locally defined in a neighborhood of the origin of R^n, so that

A1.1)

$$H_{uu}(0,0,0) > 0,$$
$$H_{ww}(0,0,0) - H_{wu}(0,0,0)H_{uu}^{-1}(0,0,0)H_{uw}(0,0,0) < 0$$

where

$$H(x,u,w) \;=\; V(A(x)+B(x)u+E(x)w) - V(x) + \frac{1}{2}(||C(x)+D(x)u+F(x)w)||^2 - \gamma^2||w||^2),$$

is Hamiltonian function associated with the control problem, and

$$\frac{\partial^2 H}{\partial(u,w)^2}(x,u,w) = \begin{bmatrix} H_{uu}(x,u,w) & H_{wu}(x,u,w) \\ H_{uw}(x,u,w) & H_{ww}(x,u,w) \end{bmatrix}$$

A1.2) The discrete Hamilton-Jacobi-Issacs Equation

$$H(x,u^*(x),w^*(x)) = 0 \tag{1.9}$$

holds in a neighborhood of $0 \in R^n$ for $u^*(x)$ and $w^*(x)$ which satisfy

$$\begin{aligned} u^*(0) &= 0 \\ w^*(0) &= 0 \\ H_u(x,u^*(x),w^*(x)) &= 0 \qquad (1.10) \\ H_w(x,u^*(x),w^*(x)) &= 0 \qquad (1.11) \end{aligned}$$

A1.3) The equilibrium $x = 0$ of the system

$$x(t+1) = A(x(t)) + B(x(t))u^*(x(t)) + E(x(t))w^*(x(t))$$

is locally asymptotically stable.

Then, the full information feedback control law

$$\begin{aligned} u &= k(x,w) \\ &= u^*(x) - H_{uu}^{-1}(x,u^*(x),w^*(x)) \cdot H_{wu}(x,u^*(x),w^*(x))(w - w^*(x)) \qquad (1.12) \end{aligned}$$

solves the problem of disturbance attenuation with internal stability.

Remark 1.1 Under assumption A1.1, the matrix $\frac{\partial^2 H}{\partial(u,w)^2}(x,u,w)$ is invertible in a neighborhood of the origin. By the implicit function theorem, there exist two smooth functions $u^*(x)$ and

$w^*(x)$ defined in a neighborhood of $0 \in R^n$ by equations (1.10) and (1.11) and satisfy $u^*(0) = 0$ and $w^*(0) = 0$.

We can see that the nonlinear discrete-time H_∞ control law (1.12) relies on the solution of (1.10) to (1.9). However, like the continuous-time case, it is also impossible to get the closed from solution for these equations except for some simple cases. In [23], an approximation approach to solving the DHJI equation is developed, which has made the discrete-time nonlinear H_∞ control a practical design tool. In this thesis, we will apply this algorithm to design an approximate H_∞ controller for the discretized RTAC system. The detailed summary of the algorithm will be given in Section 4.1.

1.5 Flight Control in Windshears

Windshears were an important reason for several civil aviation accidents in past years. This process of accidents can be roughly accounted as follows. When an aircraft takes off in a windshear, the wind boosts its head. To maintain a constant climb rate, the pilot may usually reduce the angle of attack. During the aircraft enters the core of the wind, the downdraft and decreased head wind will reduce its attack angle but the pilot does not know this point and keeps on reducing the angle of attack. Then, as the result of this, the aircraft loses its attitude rapidly and crashes when the windshear is severe.

Here we hope to improve the flight system's auto recovery ability, i.e., to minimize the gain between the windshear and the vertical acceleration of the aircraft, which, in some sense, will help the aircraft to avoid losing attitude badly.

If we consider the angle of attack as the control of aircraft, the flight control of aircraft encountering windshears after takeoff can be treated as a nonlinear H_∞ problem, i.e., to minimize the L_2 gain between the windshear and the vertical acceleration of the aircraft.

The kinematic and dynamic motions of the mass center of the aircraft are based on inertial reference frame and wind based reference frame, respectively. The model of the control problem was constructed in [26] and [27]. First of all, there are a few assumptions in [26] and [27] during the construction of the motion of aircraft: 1) during takeoff maximum thrust is used., 2) the aircraft mass is constant, 3) flight is in the vertical plane, 4) air density is constant, 5) rotational inertial of the aircraft and dynamics for actuators and sensors are not considered. Furthermore, in order to apply the approximation method developed, the kinematical and dynamical equations of motion of the aircraft with above assumptions are simplified as:

$$\begin{aligned}
m\dot{V} &= T - D - mg\gamma - m(\dot{W}_x + \dot{W}_h\gamma), \\
mV\dot{\gamma} &= T(\alpha+\delta) + L - mg + m(\dot{W}_x\gamma - \dot{W}_h), \\
\ddot{h} &= \frac{T}{m}(\gamma+\delta) - \frac{b_0\rho SV^2}{2m}\gamma + \frac{c_0\rho SV^2}{2m} - g
\end{aligned} \tag{1.13}$$

where

$$\begin{aligned}
T &= a_0 + a_1V + a_2V^2 \\
D &= \frac{1}{2}C_D\rho SV^2 = \frac{1}{2}(b_0 + b_1\alpha)\rho SV^2 \\
L &= \frac{1}{2}C_L\rho SV^2 = \frac{1}{2}(c_0 + c_1\alpha)\rho SV^2.
\end{aligned}$$

The notations used here are the same as those in [26]: D: drag force(lb); g: gravitational force per unit mass (ft sec^{-2}); h: vertical coordinate of aircraft center of mass(ft); L: lift force(lb); m: aircraft mass (lb ft^{-1} sec^{-2}); O: center of mass of aircraft; S reference surface (ft^2); t: time (sec); T: thrust force (lb); V: aircraft speed relative to wind speed frame (ft sec^{-1}); W_x: horizontal component of wind velocity (ft sec^{-1}); W_h: vertical component of wind velocity (ft sec^{-1}); α: angle of attack (rad); γ: relative path inclination (rad); δ: thrust inclination (rad); ρ: air density (lb ft^{-4} sec^2).

Here we look at V and γ as the states of the system and α as the input. And, the control objective is to design the control law so that:

$$\int_0^t (|\ddot{h}(s)|^2 + |M\alpha(s)|^2)ds \le \bar{\gamma}\int_0^t (|\dot{W}_x(s)|^2 + |\dot{W}_h(s)|^2)ds$$

for a minimum $\bar{\gamma}$. Here M is a constant. This problem can be formulated as a nonlinear H_∞ control problem. In this thesis we will find the continuous-time full information approximation nonlinear H_∞ control laws to achieve disturbance attenuation for it.

1.6 Nonlinear Benchmark System

The RTAC system, depicted in Figure 1.2, consists of a cart of mass M connected to a fixed wall by a linear spring of stiffness k. The cart is constrained to have one-dimensional travel. The proof-mass actuator attached to the cart has mass m and moment of inertia I about its center of mass, which is located at a distance e from the point about which the proof-mass rotates. Its motion occurs in a horizontal plane so that no gravitational forces need to be considered. The motion of RTAC is described in [20] and is repeated as follows

$$\begin{aligned}
\ddot{\xi} + \xi &= \epsilon(\dot{\theta}^2\sin\theta - \ddot{\theta}\cos\theta) + F \\
\ddot{\theta} &= -\epsilon\ddot{\xi}\cos\theta + u
\end{aligned} \tag{1.14}$$

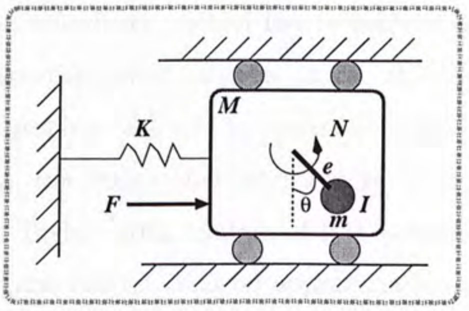

Figure 1.2: Rotational/translational actuator

where ξ is the one-dimensional displacement of the cart, θ the angular position of the proof body, w the disturbance, and u the control input. The coupling between the translational and rotational motion is captured by the parameter ϵ which is defined by

$$\epsilon = \frac{me}{\sqrt{(I+me^2)(M+m)}}$$

where $0 < \epsilon < 1$ is the eccentricity of the proof body.

Letting $x = \mathrm{col}(x_1\ x_2\ x_3\ x_4) = \mathrm{col}(\xi\ \dot{\xi}\ \theta\ \dot{\theta})$ and $y = \xi$ yields the following state space representation of (1.14):

$$\begin{aligned} \dot{x} &= f(x) + g_1(x)u + g_2(x)F \\ y &= x_1 \end{aligned} \tag{1.15}$$

where

$$f = \begin{bmatrix} x_2 \\ \frac{-x_1+\epsilon x_4^2 \sin x_3}{1-\epsilon^2\cos^2 x_3} \\ x_4 \\ \frac{\epsilon \cos x_3 (x_1 - \epsilon x_4^2 \sin x_3)}{1-\epsilon^2\cos^2 x_3} \end{bmatrix},$$

$$g_1 = \begin{bmatrix} 0 \\ \frac{-\epsilon\cos x_3}{1-\epsilon^2\cos^2 x_3} \\ 0 \\ \frac{1}{1-\epsilon^2\cos^2 x_3} \end{bmatrix}, \quad g_2 = \begin{bmatrix} 0 \\ \frac{1}{1-\epsilon^2\cos^2 x_3} \\ 0 \\ \frac{-\epsilon\cos x_3}{1-\epsilon^2\cos^2 x_3} \end{bmatrix}$$

where $1-\epsilon^2\cos^2 x_3 \neq 0$ for all x_3 and $\epsilon < 1$.

Under a mild persistent disturbance, the position of the cart will oscillate or even go unbounded. We will consider how to design a state feedback controller such that the affect of the disturbance to the position of the cart can be attenuated to a certain degree. This control problem can be formulated as a nonlinear H_∞ control problem.

This problem of designing a feedback control law to achieve asymptotic disturbance attenuation while maintaining a good transient response in the closed-loop system is known as the benchmark nonlinear control problem [20], and has been an intensive research subject since 1995. In particular, in [29] and [21], the authors designed a third order and fifth order approximate H_∞ control law based on the Taylor series solution of HJI equation, respectively. In this thesis we will further find discrete-time full information approximation nonlinear H_∞ control law to achieve disturbance attenuation for the discrete-time RTAC system.

1.7 Outline of the Work

The remaining chapters of the thesis are organized as follows.

Chapter 2: The thesis starts from attitude tracking control of spacecraft. The general framework of output regulation problem is introduced. The model of motions of spacecraft is constructed. And the general framework mentioned is applied to solve the tracking control problem in both the inertial matrix known case and the unknown case.

Chapter 3: The approximation approach to solving the HJI equation is summarized. And the results will be used to solve the disturbance attenuation problems of flight control system in windshears. Some simulations are given to evaluate the control strategies.

Chapter 4: The approximation approach to solving the DHJI equation is summarized. And the results will be used to solve the disturbance attenuation problems of the discretized RTAC system. Like continuous case, some simulations are shown to evaluate the control law.

Chapter 5: Finally, some concluding remarks and recommendations for further research are given.

In the Appendix, the programs used in Chapter 4 and 5 are partially attached.

The thesis is accompanied by many examples with numerical simulations based on MATLAB.

Chapter 2

Attitude Control and Asymptotic Disturbance Rejection of Rigid Spacecraft

This Chapter mainly discusses the attitude tracking control problem of rigid spacecraft. Here we apply the nonlinear output regulation theory to formulate the control problem and solve it with a general framework of output regulation.

The rest of chapter is organized as follows: In Section 2.1, the model of spacecraft kinematics and dynamics is introduced. In Section 2.2, the attitude tracking control problem of a rigid spacecraft is described. In Section 2.3, the general method of output regulation is presented. In Section 2.4, the control problem is formulated as a global robust output regulation problem and the general framework is applied. Numerical simulation is given in Section 2.5. Finally we close the chapter in Section 2.6 by some remarks and prospect.

2.1 Model Description

Nowadays spacecrafts, especially microsatellites play a key role in space missions, such as communication, earth observation, position location and so on. The performances of the satellites rely on the effectiveness of complex onboard control systems. In this section, we will describe its mathematical model, which have been studied in several books including [2], [12], and [13].

It is known that *reference coordinate frames* shown in Figure 2.1, and *orientation vector* are two key bases for the attitude dynamics and kinematics of spacecraft ([13], Chapter 4 and Appendix A) and ([12], Chapter 9). The attitude of a three-dimensional body is mostly defined with a set of axes fixed to the body. This set of axes, usually orthogonal coordinates, is named a body reference frame. Another useful reference frame is inertial reference frame. For an earth-orbiting spacecraft, an inertial coordinate system is defined with the center of mass of the earth as well as its direction in space is inertial with respect to the solar system. A set of orthonormal basis vectors $\{\mathbf{x_r}, \mathbf{y_r}, \mathbf{z_r}\}$, that the coordinate frame system can be expressed based on, obeys the right hand rule and can be regarded as a linear operator $F_r : R^3 \rightarrow \mathbf{V}$ (a row of three vectors) with definition $F_r = [\mathbf{x_r}, \mathbf{y_r}, \mathbf{z_r}]^{\mathbf{T}}$. The attitude operator could be regarded as a coordinate transformation that transforms a given reference frame F_r to a body fixed frame F_b. This transformation is based on the *direction cosine matrix* [13] C_{br} and has the form

$$C_{br} = F_b {F_r}^{-1} = F_b {F_r}^*, \tag{2.1}$$

where $*$ means the adjoint of a linear transform. It is easy to check that

$$\|C_{br}\| = 1 \tag{2.2}$$

and

$$C_{br}^* C_{br} = I_3$$

thanks to the orthogonality of the basis vectors.

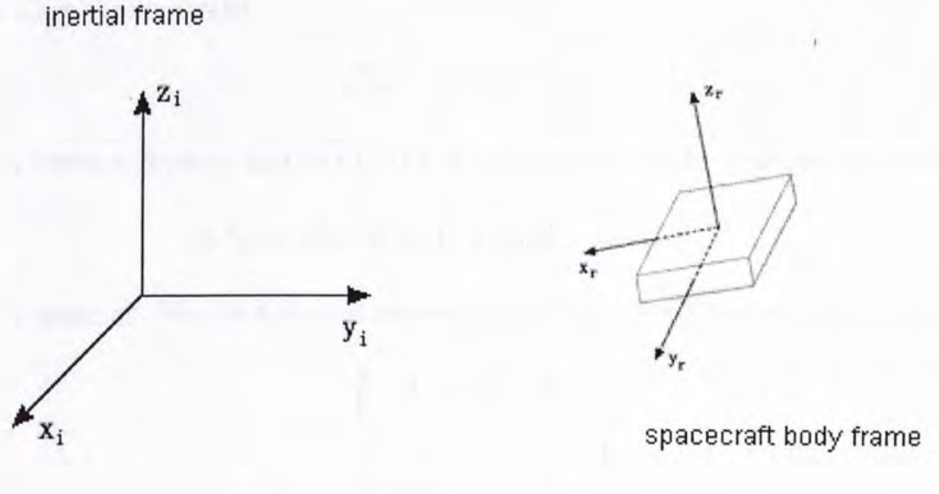

Figure 2.1: The inertial reference frame and the body reference frame.

This attitude representation C_{br} is a 3×3 matrix with 9 elements and subject to 6 constrains. We need simpler representations. Usually there are two widely used mathematical representations of spacecraft attitude. One is in term of *Euler Angle Rotation* and the other based on the *Unit Quaternion*[13]. Compared with the first method, attitude representation with the unit quaternion is free of singularity and has one constraint, though it is lack with geometric meanings. In this paper, we mainly use this representation.

The quaternion's basis definition is a consequence of the properties of the direction cosine matrix C_{br}. There are three eigenvalues for C_{br}: $\{1, e^{j\Phi}, e^{-j\Phi}\}$, i.e., this matrix has at least one eigenvector with eigenvalue of 1, and this eigenvector is unchanged by the matrix C_{br}:

$$C_{br}\mathbf{e} = 1\mathbf{e}.$$

This eigenvector $\mathbf{e} = [e_1, e_2, e_3]^T$ has the same components in the body frame or another reference frame. The existence of such an eigenvector is the analytical demonstration of Euler's famous theorem about the rotation displacement: "The most general displacement of a rigid body with one point fixed is a rotation about some axis" [13]. It is easy to see the eigenvector $\mathbf{e}$ here is just the fixed axis. In this way, the transform with matrix C_{br} in (2.1) can be achieved by a single angular rotation about $\mathbf{e}$ over an angle Φ. The quaternion is a vector defined as follows:

$$q = [q_1, q_2, q_3, q_4]^T$$
$$q_i = e_i \sin(\frac{\Phi}{2}),\ q_4 = \cos(\frac{\Phi}{2}),\ i = 1, \ldots, 3,$$

where $q_v = [q_1, q_2, q_3]^T$ is called the vector part of q and q_4 is the scalar part of q. Obviously, the quaternion q has the constraint

$$q_v^T q_v + q_4^2 = 1. \tag{2.3}$$

The relation between rotation matrix C_{br} and the quaternion can be expressed as follows [13]:

$$C_{br}(q) = (q_4^2 - q_v^T q_v)I_3 + 2q_v q_v^T - 2q_4 q_v^\times, \tag{2.4}$$

where the operator $q_v^\times$ denotes a skew symmetric matrix for a 3×1 vector q_v and has the form

$$q_v^\times = \begin{pmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{pmatrix},$$

which satisfies the following properties ([2], Appendix A): for all $a \in R^3$ and $b \in R^3$,

$$\begin{aligned} (a^\times)^T = -a^\times,\ a^\times b = -b^\times a,\ a^\times a = 0, \\ a^T b^\times a = 0,\ (a+b)^\times = a^\times + b^\times,\ a^\times b^\times = ba^T - a^T b I_3, \\ (a^\times b)^\times = ba^T - ab^T,\ \|a^\times\| = \|a\| = \sqrt{a^T a}. \end{aligned} \tag{2.5}$$

Differentiating q, it is easy to obtain the attitude kinematics in terms of the unite quaternion [12][2][13] [5][4]:

$$\begin{aligned}\dot{q}_v &= 0.5(q_4 I_3 + q_v^{\times})\Omega_{br} \\ \dot{q}_4 &= -0.5 q_v^T \Omega_{br}\end{aligned} \tag{2.6}$$

where Ω_{br} is the angular velocity of the body frame F_b with respect to a reference frame F_r and is expressed in the frame F_b.

Another important result in attitude representation with the unit quaternion is what comes after two successive angular transformations, i.e., how to express q'' in terms of q' and q if $q'' = [q_v''^T, q_4'']^T$ is obtained first from $q = [q_v^T, q_4]^T$ and then from $q' = [q_v'^T, q_4']^T$, two rotations. The final transformation can be achieved using rotation matrix representation,

$$C(q'') = C(q')C(q).$$

From this equation, or from a direct quaternion multiplication, we could get the expression of q'' [2], shown as follows:

$$\begin{aligned}q_v'' &= [q_4' I_3 - (q_v')^{\times}]q_v + q_4 q_v' \\ q_4'' &= -(q_v')^T q_v + q_4 q_v'\end{aligned} \tag{2.7}$$

A typical spacecraft structure usually consists of two basic parts: the rigid center body and the flexible solar arrays. Here we could look at the structure just as a rigid body and the solar arrays, onboard payload motion, and fuel consumption as the uncertainties of system. Based on Euler's moment equation [13], we reach the dynamic equation of attitude motion [4]:

$$J\dot{\Omega} = -\Omega^{\times} J\Omega + u + d \tag{2.8}$$

where

$$J = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{12} & J_{22} & J_{23} \\ J_{13} & J_{21} & J_{33} \end{pmatrix}$$

is the positive definite, overall inertia matrix of the integrated satellite system , $u \in R^3$ is the control torques, and $d \in R^3$ is the external disturbance, including gravity-gradient torque, aerodynamic torque, residual magnetic torque and solar radiation torque. Here we also assume the spacecraft is equipped with three independent actuators which are able to provide three independent control torques at each time instant [4].

2.2 Problem Formulation

In this section, we will formulate the main control problem, i.e., to achieve attitude tracking and input disturbance rejection of solid spacecraft.

From the description in the second section, we combine the equations (2.6) and (2.8) and come to the spacecraft system modelled,

$$\begin{aligned}\dot{q}_v &= 0.5(q_4 I_3 + q_v^{\times})\Omega \\ \dot{q}_4 &= -0.5 q_v^T \Omega \\ J\dot{\Omega} &= -\Omega^{\times} J\Omega + u + d,\end{aligned} \tag{2.9}$$

where we assume external disturbance d to be the combination of some sinusoidal functions without either amplitudes or phases known.

The desired attitude motion is modelled by

$$\begin{aligned}\dot{q}_{dv} &= 0.5(q_{d4} I_3 + q_{dv}^{\times})\Omega_d \\ \dot{q}_{d4} &= -0.5 q_{dv}^T \Omega_d,\end{aligned} \tag{2.10}$$

where $\mathrm{col}(q_{dv}, q_{d4}) \in R^3 \times R$ is the unit quaternion satisfying

$$q_{dv}^T q_{dv} + q_{d4}^2 = 1$$

representing the target attitude, and $\Omega_d \in R^3$ is the target angular velocity in the body frame F_r with respect to the inertial frame F_i and expressed in the frame F_r. This desired attitude motion model can be found in many literatures on attitude tracking problem, e.g., [5].

Now, the control objective of the paper is to design a controller, independent of d, such that, the states of the closed-loop system are bounded, and

$$\lim_{t\to\infty} \mathrm{col}(q_v(t) - q_{dv}(t), q_4(t) - q_{d4}(t)) = 0. \tag{2.11}$$

The error rotation between actual attitude motion and steady attitude motion can be also addressed with a new unite quaternion $e = [e_v^T, e_4]$ [2][5], where

$$e_v^T e_v + e_4^2 = 1. \tag{2.12}$$

Based on the equation (2.7) and $C(e) = C(q)C(q_c)^{-1}$, the coordinate transformation can be expressed as follows:

$$\begin{aligned}e_v &= q_{d4} q_v - q_{dv}^{\times} q_v - q_4 q_{dv} \\ e_4 &= q_{dv}^T q_v + q_4 q_{d4} \\ \omega &= \Omega - C\Omega_d,\end{aligned} \tag{2.13}$$

where, ω is the error angular velocity in the frame F_b with respect to the frame F_r, and

$$C = (1 - 2e_v^T e_v)I_3 + 2e_v e_v^T - 2e_4 e_v^\times$$

based on the equations (2.3) and (2.4).

Then we have

$$\begin{aligned} \dot{e}_v &= 0.5(e_4 I_3 + e_v^\times)\omega \\ \dot{e}_4 &= -0.5e_v^T \omega \end{aligned} \tag{2.14}$$

from the equation (2.6). Moreover, the third equation of (2.9) in the new coordinate becomes

$$J\dot{\omega} = -(\omega + C\Omega_d)^\times J(\omega + C\Omega_d) + J(\omega^\times C\Omega_d - C\dot{\Omega}_d) + u + d. \tag{2.15}$$

It have been proved in [14] that the equation (2.11) holds if

$$\lim_{t\to\infty} e_v(t) = 0. \tag{2.16}$$

To this end, when $e_v(t) = 0$, $C(q_e) = C = I_3$, which means $I_3 = C(q)C(q_c)^{-1}$ and $C(q) = C(q_c)$, i.e., $q_v(t) - q_{dv}(t) = 0$ and $q_4(t) - q_{d4}(t) = 0$.

The control objective (2.11) or (2.16) was achieved in [1] [2] [3] [5] when zero external disturbance were considered. In [4], H_∞ nonlinear control law was studied to achieve disturbance attenuation, but the error rotation could not approach 0 when $t \to \infty$ for disturbances with unbounded energy. In the rest, we will use the output regulation theory to formulate the problem and achieve the control objective (2.16) as well as unbounded energy disturbance asymptotic rejection in the first time.

2.3 Preliminaries of General Framework for Global Robust Output Regulation

In this section, we will summarize the general framework for the output regulation problem, which includes three steps: First, solve the regulator equations; Second, find an appropriate steady-state generator for the system and design a generalized internal model based on the generator; Third, after some coordinate and input transformation, we convert the global robust output regulation problem into a global robust stabilization problem and solve the stabilization problem. The following result is from [7]:

Consider a plant described by

$$\begin{aligned} \dot{x} &= f(x, u, v, w),\ x(0) = x_0 \\ y &= h(x, u, v, w),\ t \geq 0 \end{aligned} \tag{2.17}$$

and an exosystem described by

$$\dot{v} = a(v), \; v(0) = v_0 \tag{2.18}$$

where x is the n-dimensional plant state, u the m-dimensional plant input, y the p-dimensional plant output representing the tracking error, v the q-dimensional exogenous signal representing the disturbance and/or the reference input, and w the N-dimensional plant uncertain parameter whose nominal value is 0. The functions f, h and a are sufficiently smooth satisfying $f(0,0,0,w) = 0$ and $h(0,0,0,w) = 0$ for all w, and $a(0) = 0$.

Let us first list two standard assumptions.

A2.1: The equilibrium of the exosystem (2.18) at $v = 0$ is stable.

A2.2: There exist sufficiently smooth functions $\mathbf{x}(v,w)$ and $\mathbf{u}(v,w)$ with $\mathbf{x}(0,0) = 0$ and $\mathbf{u}(0,0) = 0$ satisfying, for all $v \in V$, and $w \in W$ where V is an open neighborhood of the origin of R^q and W an open neighborhood of the origin of R^N, the following equations

$$\begin{aligned} \frac{\partial \mathbf{x}(v,w)}{\partial v} a(v) &= f(\mathbf{x}(v,w), \mathbf{u}(v,w), v, w) \\ 0 &= h(\mathbf{x}(v,w), \mathbf{u}(v,w), v, w). \end{aligned} \tag{2.19}$$

Definition 2.1 Let $g : R^{n+m} \mapsto R^l$ be a mapping where $1 \leq l \leq n+m$. Under Assumptions A1 and A2, the nonlinear system (2.17) and (2.18) is said to have a *steady state generator* with output $g(x,u)$ if there exists a triple $\{\theta, \alpha, \beta\}$, where $\theta : R^{q+N} \mapsto R^s$, $\alpha : R^s \mapsto R^s$, and $\beta : R^s \mapsto R^l$ for some integer s are sufficiently smooth functions vanishing at the origin, such that, for all trajectories $v(t) \in V$ of (2.18) and all $w \in W$,

$$\begin{aligned} \frac{d\theta\left(v\left(t\right), w\right)}{dt} &= \alpha\left(\theta\left(v\left(t\right), w\right)\right) \\ g(\mathbf{x}(v(t), w), \mathbf{u}(v(t), w)) &= \beta\left(\theta\left(v\left(t\right), w\right)\right). \end{aligned} \tag{2.20}$$

If, in addition, the linearization of the pair $\{\beta(\theta), \alpha(\theta)\}$ at the origin is observable, then $\{\theta, \alpha, \beta\}$ is called a linearly observable steady state generator with output $g(x,u)$. ∎

In the following, we assume $g(x,u) = [x_{i_1}, x_{i_2}, \ldots, x_{i_d}, u]^T$ where $1 \leq i_1 \leq i_2 \leq \cdots \leq i_d \leq n$ and $0 \leq d \leq n$. The definition of the steady state generator leads to a general characterization of the internal model as follows.

Definition 2.2 Under assumptions A1 and A2, suppose the system (2.17) and (2.18) has a steady state generator with output $g(x,u)$. Let $\gamma : R^{s+n+m} \mapsto R^s$ be a sufficiently smooth function vanishing at the origin. Then we call the following system

$$\dot{\eta} = \gamma\left(\eta, x, u\right) \tag{2.21}$$

an *internal model* with output $g(x, u)$ if, for all trajectories $v(t) \in V$ of (2.18) and all $w \in W$,

$$\gamma\left(\theta\left(v\left(t\right), w\right), \mathbf{x}\left(v\left(t\right), w\right), \mathbf{u}\left(v\left(t\right), w\right)\right) = \alpha\left(\theta\left(v\left(t\right), w\right)\right). \tag{2.22}$$

∎

In [7], Huang and Chen also presented under what conditions the steady state generator and proper internal model defined above exist, i.e., under what conditions, the global robust output regulation problem for the plant (2.17) and (2.18) can be converted into a global robust stabilization problem for an augmented system. In the following we will highlight those conditions.

A2.3: The exosystem (2.18) is linear, i.e., $a(v) = A_1 v$, for some matrix A_a, and all the eigenvalues of A_1 are simple with zero real parts.

Definition 2.3 Let $\pi(v(t), w)$ be trigonometric polynomial of t or polynomial in $v(t)$. Let $P(\lambda) = \lambda^r - a_1 - a_2\lambda - \cdots - a_r\lambda^{r-1}$ be a monic polynomial in λ, $P(\lambda)$ is called a *zeroing polynomial* of $\pi(v, w)$ if, along all trajectories $v(t)$ of the exosystem, and $w \in R^N$,

$$\frac{d^r\pi(v, w)}{dt^r} - a_1\pi(v, w) - a_2\frac{d\pi(v, w)}{dt} - \cdots - a_r\frac{d^{r-1}\pi(v, w)}{dt^{r-1}} = 0. \tag{2.23}$$

$P(\lambda)$ is called a *minimal zeroing polynomial* of $\pi(v, w)$ if $P(\lambda)$ is a zeroing polynomial of $\pi(v, w)$ of least degree. ∎

Definition 2.4 Let $\pi_i(v(t), w), i = 1, \ldots, I$, for some positive integer I, be I trigonometric polynomials of t or polynomials in $v(t)$. They are called *pairwise coprime* if their minimal zeroing polynomials $P_1(\lambda), \ldots, P_I(\lambda)$ are pairwise coprime. ∎

Under Assumption $A2.2$ and $A2.3$, suppose, for $i = 1, \ldots, d+m$, there exist pairwise coprime polynomials $\pi_i^1(v(t), w), \ldots, \pi_i^{I_i}(v(t), w)$ with $r_i^1, \ldots, r_i^{I_i}$ being the degrees of their minimal zeroing polynomials $P_i^1(\lambda), \ldots, P_i^{I_i}(\lambda)$, and a sufficiently smooth function $\Gamma_i : R^{r_i} \longmapsto R$ with $r_i = r_i^1 + \cdots + r_i^{I_i}$ vanishing at the origin such that, for all trajectories $v(t)$ of the exosystem, and $w \in R^N$, the solutions of regulator equations satisfy:

$$\begin{aligned} g_i(\mathbf{x}(v, w), \mathbf{u}(v, w)) = \Gamma_i(&\pi_i^1(v, w), \dot{\pi}_i^1(v, w), \ldots, \frac{d^{(r_i^1-1)}\pi_i^1(v, w)}{dt^{(r_i^1-1)}}, \\ &\ldots, \pi_i^{I_i}(v, w), \dot{\pi}_i^{I_i}(v, w), \ldots, \frac{d^{(r_i^{I_i}-1)}\pi_i^{I_i}(v, w)}{dt^{(r_i^{I_i}-1)}}). \end{aligned} \tag{2.24}$$

Then the system (2.17) and (2.18) has a well defined steady-state generator with output $g(x, u)$. Based on this steady-state generator, it is possible to find a proper internal model.

Attaching the internal model to the given plant yields the following augmented system

$$\dot{x} = f(x,u,v,w),\ \dot{\eta} = \gamma(\eta, x, u),\ y = h(x,u,v,w). \tag{2.25}$$

Performing on (2.25) the following coordinate and input transformation

$$\begin{aligned} \bar{x}_i &= x_i - \beta_i(\eta),\ i = 1,\cdots,d \\ \bar{x}_i &= x_i - \mathbf{x}_i(v,w),\ i = d+1,\cdots,n \\ \bar{\eta} &= \eta - \theta(v,w) \\ \bar{u} &= u - \beta_u(\eta) = u - [\beta_{d+1}(\eta),\cdots,\beta_{d+m}(\eta)]^T \end{aligned} \tag{2.26}$$

gives a new system denoted by

$$\dot{\bar{x}} = \bar{f}(\bar{x},\bar{\eta},\bar{u},v,w),\ \dot{\bar{\eta}} = \bar{\gamma}(\bar{x},\bar{\eta},\bar{u},v,w),\ y = \bar{h}(\bar{x},\bar{\eta},v,w) \tag{2.27}$$

where $\bar{x} = \mathrm{col}(\bar{x}_1,\cdots,\bar{x}_n)$. It can be verified [7] that the system has the property

$$\begin{aligned} \bar{f}(0,0,0,v,w) &= 0 \\ \bar{\gamma}(0,0,0,v,w) &= 0 \\ \bar{h}(0,0,0,v,w) &= 0. \end{aligned} \tag{2.28}$$

Then if a controller of the form

$$\begin{aligned} \bar{u} &= k(\bar{x}_1,\cdots,\bar{x}_d,\xi) \\ \dot{\xi} &= g_\xi(\bar{x}_1,\cdots,\bar{x}_d,\xi,y) \end{aligned} \tag{2.29}$$

where $\xi \in R^{n_\xi}$ for some integer n_ξ, and k and g_ξ are sufficiently smooth functions vanishing at their respective origins stabilizes the equilibrium point $(\bar{x},\bar{\eta}) = (0,0)$ of (2.27), then the following controller

$$\begin{aligned} u &= \beta_u(\eta) + k(x_1 - \beta_1(\eta),\cdots,x_d - \beta_d(\eta),\xi) \\ \dot{\eta} &= \gamma(x,\eta,u) \\ \dot{\xi} &= g_\xi(x_1 - \beta_1(\eta),\cdots,x_d - \beta_d(\eta),\xi,y) \end{aligned} \tag{2.30}$$

solves the robust output regulation problem for the original plant (2.17) and the exosystem (2.18).

2.4 Application of Global Robust Output Regulation

In this section, we will apply output regulation theory to formulate the control problem raised in section 2.2 and solve it with the framework just mentioned. Two cases are studied, without or with system uncertainty, separately. Actually, in the first case, without system uncertainty, it is not necessary to apply an internal model to design the controller, and a simple observer is enough. However, in order to keep the continuity and to better demonstrate the framework, among both cases, internal models are used.

2.4.1 Case I: without unknown parameters

In this subsection, we suppose there is no unknown parameter in the plant, i.e., J are totally known.

In order to make $\lim_{t\to\infty} e_v(t) = 0$, it suffices to choose

$$\omega = -Ke_v, \tag{2.31}$$

where $K \in R^{3\times 3}$ is positive definite and symmetric. It has been proved in [5] and we repeat it here for self-containiness. First we show that $e_4 \to 1$ under this controller as $t \to \infty$ whenever $e_v(0) \neq -1$. Let k_1 and k_2 be the minimum and maximum eigenvalues of K. We have

$$0.5k_1(1-e_4^2) \leq \dot{e}_4 = 0.5e_v^TKe_v \leq 0.5k_2(1-e_4^2),$$

from the second equation of (2.14) and the condition $e_v^Te_v + e_4^2 = 1$. By the comparison principle ([9], lemma 2.5, pp.85), e_4 satisfies the following inequalities:

$$1 - \frac{2[1-e_4(0)]e^{(-k_1t)}}{1+e_4(0)+[1-e_4(0)]e^{(-k_1t)}} \leq e_4(t) \leq 1 - \frac{2[1-e_4(0)]e^{(-k_2t)}}{1+e_4(0)+[1-e_4(0)]e^{(-k_2t)}}$$

for all $t \geq 0$. If $e_4(0) = -1$, $e_v(0) = 0$ and then $e_4(t) \equiv -1$ for all $t \geq 0$. When $e_4(0) \neq -1$, $e_4(t) \geq -1$ for all $t \geq 0$ and $\lim_{t\to\infty} e_4(t) = 1$, i.e., $\lim_{t\to\infty} e_v(t) = 0$. We also could find a Lyapunov function for the close-loop system (2.14) and (2.31) to prove the global asymptotic stability of $e_v = 0$,

$$V = ce_v^Te_v + c(1-e_4)^2,$$

where the constant $c \geq 0$. The derivative of V is given by

$$\dot{V} = ce_v^T\omega = -ce_v^TKe_v \leq -ck_1|e_v|^2 \leq 0.$$

By LaSalle's theorem ([9], Theorem 3.4, pp 115), the global asymptotic stability of $e_v = 0$ stands for all initial conditions $e_v(0)$. On the other hand, both $(e_v, e_4) = (0,-1)$ and $(e_v, e_4) = (0,1)$ have the same physical meaning, we finish the proof.

For simplicity, in the remaining part, we only consider the problem of designing u to make ω converge to $-Ke_v$. To this end, let $z = \omega + Ke_v$, and make the state z of the system

$$\begin{aligned} J\dot{z} = & - (\omega + C\Omega_d)^\times J(\omega + C\Omega_d) + J(\omega^\times C\Omega_d - C\dot{\Omega}_d) \\ & + JK\left[0.5(e_4 I_3 + e_v^\times)\omega\right] + u + d \end{aligned} \tag{2.32}$$

converge to 0. We will appeal to the output regulation theory to achieve this control objective.

Since the variables ω, Ω_d, e_v and e_4 are known or measurable, we first apply a pre-compensator

$$u = u' + (\omega + C\Omega_d)^\times J(\omega + C\Omega_d) - J(\omega^\times C\Omega_d - C\dot{\Omega}_d) - JK\left[0.5(e_4 I_3 + e_v^\times)\omega\right]$$

to system (2.15) and (2.14). Meanwhile, suppose the disturbance is generated by some autonomous system through

$$d(t) = \mathbf{d}(v_d(t), w_d),$$

where $\mathbf{d} : R^{n_d+N} \mapsto R^3$ is some sufficiently smooth function, $w_d \in R^N$ is the unknown parameter, and $v_d(t) \in R^{n_d}$ is the exogenous signal produced by

$$\dot{v}_d = A_d v_d. \tag{2.33}$$

In particular, v_d produced by (2.33) is the combination of some sinusoidal functions without either amplitudes or phases known.

Then the system formulated with output regulation theory can be written as

$$\begin{aligned} J\dot{z} &= u' + d(v_d, w_d) \\ \dot{v}_d &= A_d v_d \\ y &= z. \end{aligned} \tag{2.34}$$

Here, the dynamics (2.33) governing v_d is called exosystem.

In order to solve the global output regulation problem of the nonlinear system (2.34), we first need to convert the global output regulation problem to a global stabilization problem. Formulate (2.34) as the form of system (2.17) and (2.18) with,

$$\begin{aligned} x &= z,\ u = u',\ v = v_d,\ y = z,\ w = w_d, \\ f(x,u,v,w) &= J^{-1}(u' + d) \\ h(x,u,v,w) &= z, \\ a(v) &= A_d v. \end{aligned} \tag{2.35}$$

For this plant, just as cited in Section 2.2, if Assumptions $A2.2$ and $A2.3$ are satisfied, and if the solutions of regulator equations satisfy equation (2.24) with some output $g(x,u)$, we could find the proper steady-state generator and then design the internal model as we have shown in Section 2.2. Attaching the internal model to the original plant yields a augmented system, and after a suitable coordinate and input transform, the global output regulation problem of the original plant can be converted to global stabilization problem of the augmented system in new coordinates and inputs.

To apply the framework, we assume:

A2.4: Suppose $\mathbf{d}_i(v_d(t), w_d)$, $i = 1, \ldots, 3$, the component of $\mathbf{d}(v_d(t), w_d)$, is a polynomial of v_d of known degree r_i and v_d satisfies (2.33).

Then there exists

$$\frac{d^{r_i}\mathbf{d}_i}{dt^{r_i}} - a_1\mathbf{d}_i - a_2\frac{d\mathbf{d}_i}{dt} - \cdots - a_r\frac{d^{(r_i-1)}\mathbf{d}_i}{dt^{(r_i-1)}} = 0.$$

Since the regulator equations of (2.34) are solvable with

$$\begin{aligned} \mathbf{x}(v,w) &= 0_{3\times1}, \\ \mathbf{u}(v,w) &= \mathbf{u}'(v,w) = -\mathbf{d}(v_d, w_d), \end{aligned} \tag{2.36}$$

and $a(v)$ is linear, the assumptions $A2.2$ and $A2.3$ are satisfied here. With the assumption $A2.4$, it is easy to check the solutions of regulator equations satisfy equation (2.24) with the output

$$g(x,u) = u'.$$

Let

$$\theta_i^1(v,w) = \begin{pmatrix} \mathbf{u}_i'(v,w) \\ \dot{\mathbf{u}}_i'(v,w) \\ \cdot \\ \cdot \\ \cdot \\ \frac{d^{(r_i'-1)}\mathbf{u}_i'(v,w)}{dt^{(r_i'-1)}} \end{pmatrix}, \ \theta_i(v,w) = T_i\theta_i^1(v,w),$$

where T_i is any suitable nonsingular matrix. We have

$$\dot{\theta}_i^1(v,w) = \Phi_i\theta_i^1(v,w)$$

for some known matrix Φ_i.

The steady-state generator with output u' can be written as

$$\dot{\theta} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} \alpha_1(\theta_1) \\ \alpha_2(\theta_2) \\ \alpha_3(\theta_3) \end{bmatrix} = \begin{bmatrix} T_1\Phi_1T_1^{-1}\theta_1 \\ T_2\Phi_2T_2^{-1}\theta_2 \\ T_3\Phi_3T_3^{-1}\theta_3 \end{bmatrix}$$

$$g(\mathbf{x}, \mathbf{u}) = \beta(\theta) = \begin{bmatrix} T_1^{-1}\theta_1 \\ T_2^{-1}\theta_2 \\ T_3^{-1}\theta_3 \end{bmatrix}.$$

To design the internal model, choose controllable pair (M_i, N_i) with M_i Hurwitz, choose T_i in the steady-state generator, so that

$$T_i\Phi_i - M_iT_i = N_i$$

and then let

$$\dot{\eta} = M\eta + Nu' \tag{2.37}$$

which is the candidate internal model with $M = \text{block diag}(M_1, M_2, M_3)$, $N = \text{block diag}(N_1, N_2, N_3)$, and $\Phi = \text{block diag}(\Phi_1, \Phi_2, \Phi_3)$. Check the condition (2.22), it is easy to find

$$M\theta(v, w) + N\mathbf{u}'(v, w) = \alpha(\theta(v, w)),$$

which proves (2.37) is a proper internal model. Attaching it to the plant (2.34) yields the augmented system.

Next, performing on the augmented system the following coordinate and input transformation: $\bar{\eta} = \eta - \theta - NJz$, $\bar{u}' = u' - \beta(\eta)$, and $\bar{z} = z$, which gives

$$\begin{aligned}
\dot{\bar{\eta}} &= M\eta + Nu' - T\Phi T^{-1}\theta - NJ\dot{z} \\
&= M\bar{\eta} + M\theta + MNJz + Nu' - T\Phi T^{-1}\theta - Nu' - Nd \\
&= M\bar{\eta} + MNJz, \qquad (2.38) \\
J\dot{\bar{z}} &= J\dot{z} = T^{-1}\theta + u' \\
&= T^{-1}\bar{\eta} + T^{-1}NJz + \bar{u}'. \qquad (2.39)
\end{aligned}$$

By now, we have converted the global output regulation problem of system (2.34) to the global stabilization problem of the new system (2.38) and (2.39). For the rest, we design a controller $\bar{u}'$, so that the new system is globally asymptotically stable.

Let

$$\bar{u}' = -K_z z, \tag{2.40}$$

where $K_z \in R^{3\times 3}$ is positive definite and symmetric and to be determined later.

Choose

$$V = 0.5\bar{\eta}^T\bar{\eta} + 0.5z^T Jz, \tag{2.41}$$

and its derivative along the close-loop system (2.38), (2.39) and (2.40) is:

$$\begin{aligned}
\frac{dV(z,\bar{\eta})}{dt} &= \bar{\eta}^T M\bar{\eta} + \bar{\eta}^T MNJz + z^T T^{-1}\bar{\eta} + z^T T^{-1}NJz - z^T K_z z \\
&= \bar{\eta}^T M\bar{\eta} + \bar{\eta}^T (MNJ + T^{-T})z + z^T T^{-1}NJz - z^T K_z z \\
&\leq -k_3\|\bar{\eta}\|^2 + \frac{\|MNJ + T^{-T}\|}{2\varepsilon}\|\bar{\eta}\|^2 + \frac{\varepsilon\|MNJ + T^{-T}\|}{2}\|z\|^2 \\
&+ \|T^{-1}NJ\|\|z\|^2 - k_z\|z\|^2,
\end{aligned} \tag{2.42}$$

where $k_3 > 0$ exists since M is Hurwitz and $k_z > 0$ is the minimum eigenvalue of K_z.

Choose a positive constant ε so that

$$\varepsilon > \frac{\|MNJ + T^{-T}\|}{k_3},$$

and choose K_z so that

$$k_z > \varepsilon\|MNJ + T^{-T}\| + 2\|T^{-1}NJ\|.$$

Then from (2.42) we have

$$\begin{aligned}
\frac{dV(z,\bar{\eta})}{dt} &\leq (-k_3 + 0.5k_3)\|\bar{\eta}\|^2 + (0.5k_z - k_z)\|z\|^2 \\
&\leq -0.5k_3\|\bar{\eta}\|^2 - 0.5k_z\|z\|^2.
\end{aligned}$$

By LaSalle Theorem, the equilibrium point at the origin of the closed-loop system (2.38), (2.39) and (2.40) is globally asymptotic stable.

Overall, the controller

$$\begin{aligned}
u &= -K_z z + T^{-1}\eta + (\omega + C\Omega_d)^\times J(\omega + C\Omega_d) \\
&\quad -J(\omega^\times C\Omega_d - C\dot{\Omega}_d) - JK\left[0.5(e_4 I_3 + e_v^\times)\omega\right] \\
\dot{\eta} &= M\eta - NK_z z + NT^{-1}\eta
\end{aligned} \tag{2.43}$$

make the state z of the system (2.32) converge to 0. Then the condition (2.11) is satisfied, and the global output regulation problem for the composite system (2.9), (2.33) and (2.10) is solved.

2.4.2 Case II: with unknown parameters

When there exist unknown parameters, i.e., J is not totally known. We assume $J = J_0 + J_w$, with J_0 known and

$$J_w = \begin{bmatrix} w_1 & w_2 & w_3 \\ w_2 & w_4 & w_5 \\ w_3 & w_5 & w_6 \end{bmatrix}$$

unknown but bounded.

In order to formulate the problem with output regulation theory, we suppose the disturbance d is generated by the same autonomous systems with Case I. Moreover, in order to design an appropriate internal model, we also suppose Ω_{di}, the components of Ω_d, are some sinusoidal functions generated by the system

$$\dot{v}_{\Omega i} = A_{\Omega i} v_{\Omega i}, \;\; t \geq 0, \;\; v_{\Omega i}(0) = v_{\Omega i0} \tag{2.44}$$

with

$$v_{\Omega i} = \begin{bmatrix} v_{\Omega i1} \\ v_{\Omega i2} \end{bmatrix} = \begin{bmatrix} \Omega_{di} \\ \frac{\dot{\Omega}_{di}}{\omega_i} \end{bmatrix}, A_{\Omega i} = \begin{bmatrix} 0 & a_i \\ -a_i & 0 \end{bmatrix}, i = 1, \ldots, 3,$$

where $v_{\Omega i0}$ and a_i are known and $v_\Omega = [v_{\Omega 1}, v_{\Omega 2}, v_{\Omega 3}]^T$.

Applying a pre-compensator

$$u = u' + (\omega + C\Omega_d)^\times J_0(\omega + C\Omega_d) - J_0(\omega^\times C\Omega_d - C\dot{\Omega}_d) - J_0 K\left[0.5(e_4 I_3 + e_v^\times)\omega\right]$$

to system (2.15) and (2.14) and letting $z = \omega + Ke_v$ gives:

$$\begin{aligned} J\dot{z} &= -(z - Ke_v + C\Omega_d)^\times J_w(z - Ke_v + C\Omega_d) + J_w((z - Ke_v)^\times C\Omega_d - C\dot{\Omega}_d) \\ &+ J_w K\left[0.5(e_4 I_3 + e_v^\times)(z - Ke_v)\right] + u' + d \\ \dot{e}_v &= 0.5(e_v I_3 + e_v^\times)(z - Ke_v) \\ \dot{e}_4 &= -0.5 e_v^T (z - Ke_v) \\ y &= z. \end{aligned} \tag{2.45}$$

Here, devote $v = \mathrm{col}(v_d, v_\Omega)$, and the dynamics (2.33) and (2.44) governing v is called *exosystem*.

In the following, we convert the global robust output regulation problem to a global robust stabilization problem. Notice here the equilibrium point of (2.45) is not in the origin, we introduce

$\bar{e}_4 = e_4 - 1$ and then formulate (2.45) as the form of system (2.17) and (2.18) with,

$$
\begin{aligned}
x &= \begin{bmatrix} z \\ e_v \\ \bar{e}_4 \end{bmatrix},\ u = u',\ v = \begin{bmatrix} v_d \\ v_\Omega \end{bmatrix},\ y = z,\ w = \begin{bmatrix} J_w \\ w_d \end{bmatrix}, \\
f(x,u,v,w) &= \begin{pmatrix} J^{-1} \begin{bmatrix} -(z - Ke_v + C\Omega_d)^\times J_w (z - Ke_v + C\Omega_d) \\ +J_w((z - Ke_v)^\times C\Omega_d - C\dot{\Omega}_d) \\ +J_w K \left[0.5(e_4 I_3 + e_v^\times)(z - Ke_v)\right] + u' + d \end{bmatrix} \\ 0.5\left[(\bar{e}_4 + 1)I_3 + e_v^\times\right](z - Ke_v) \\ -0.5 e_v^T (z - Ke_v) \end{pmatrix}, \\
h(x,u,v,w) &= e_v, \\
a(v) &= \text{block diag}(A_d, A_{\Omega_1}, A_{\Omega_2}, A_{\Omega_3})v.
\end{aligned}
$$

Since the regulator equations of (2.45) are solvable with

$$
\begin{aligned}
\mathbf{x}(v,w) &= \begin{bmatrix} 0_{3\times1} \\ 0_{3\times1} \\ 0 \end{bmatrix}, \\
\mathbf{u}(v,w) &= \mathbf{u}'(v,w) = -\mathbf{d}(v_d, w_d) + \Omega_d(v_\Omega)^\times J_w \Omega_d(v_\Omega) + J_w \dot{\Omega}_d(v_\Omega),
\end{aligned}
$$

where the condition $C = I_3$ when $e_v = 0$ is used, and $a(v)$ is linear, the assumptions $A2.2$ and $A2.3$ are satisfied here. With the assumption $A2.4$ and the assumption (2.44), it is easy to check the solutions of regulator equations satisfy equation (2.24) with the output

$$
g(x,u) = \begin{bmatrix} e_v \\ \bar{e}_4 \\ u' \end{bmatrix}.
$$

The reason we choose such an output for steady-state generator is so that we could design a full state feedback controller for the augmented system later.

In fact,

$$
g(\mathbf{x},\mathbf{u}) = \begin{bmatrix} 0_{3\times1} \\ 0 \\ \mathbf{u}' \end{bmatrix},
$$

here, the component of $\mathbf{u}'$ can be expressed as

$$
\mathbf{u}'_i(v,w) = -\mathbf{d}_i(v_d, w_d) + \mathbf{\Omega}_{di}(v_\Omega, J_w), i = 1, \ldots, 3, \tag{2.46}
$$

and, $\mathbf{\Omega}_{di}$ is polynomial of v_Ω with coefficients depending on J_w,

$$\begin{aligned}
\mathbf{\Omega}_{d1} &= -v_{\Omega_{31}}(w_2 v_{\Omega_{11}} + w_4 v_{\Omega_{21}} + w_5 v_{\Omega_{31}}) + v_{\Omega_{21}}(w_3 v_{\Omega_{11}} + w_5 v_{\Omega_{21}} + w_6 v_{\Omega_{31}}) \\
&+ w_1 v_{\Omega_{12}} a_1 + w_2 v_{\Omega_{22}} a_2 + w_3 v_{\Omega_{32}} a_3, \\
\mathbf{\Omega}_{d2} &= v_{\Omega_{31}}(w_1 v_{\Omega_{11}} + w_2 v_{\Omega_{21}} + w_3 v_{\Omega_{31}}) - v_{\Omega_{11}}(w_3 v_{\Omega_{11}} + w_5 v_{\Omega_{21}} + w_6 v_{\Omega_{31}}) \\
&+ w_2 v_{\Omega_{12}} a_1 + w_4 v_{\Omega_{22}} a_2 + w_5 v_{\Omega_{32}} a_3, \\
\mathbf{\Omega}_{d3} &= -v_{\Omega_{21}}(w_1 v_{\Omega_{11}} + w_2 v_{\Omega_{21}} + w_3 v_{\Omega_{31}}) + v_{\Omega_{11}}(w_2 v_{\Omega_{11}} + w_4 v_{\Omega_{21}} + w_5 v_{\Omega_{31}}) \\
&+ w_3 v_{\Omega_{12}} a_1 + w_5 v_{\Omega_{22}} a_2 + w_6 v_{\Omega_{32}} a_3.
\end{aligned}$$

We could look at the right side of equation (2.46) as polynomial of v of known degree r'_i, which satisfies

$$\frac{d^{r'_i}\mathbf{u}'_i}{dt^{r'_i}} - a_1\mathbf{u}'_i - a_2\frac{d\mathbf{u}'_i}{dt} - \cdots - a_r\frac{d^{(r'_i-1)}\mathbf{u}'_i}{dt^{(r'_i-1)}} = 0.$$

Let

$$\theta_i^1(v,w) = \begin{pmatrix} \mathbf{u}'_i(v,w) \\ \dot{\mathbf{u}}'_i(v,w) \\ \cdot \\ \cdot \\ \cdot \\ \frac{d^{(r'_i-1)}\mathbf{u}'_i(v,w)}{dt^{(r'_i-1)}} \end{pmatrix}, \ \theta_i(v,w) = T_i\theta_i^1(v,w),$$

where T_i is any suitable nonsingular matrix. We have

$$\dot{\theta}_i^1(v,w) = \Phi_i\theta_i^1(v,w)$$

for some known matrix Φ_i.

The steady-state generator with output $[\omega, \bar{e}_4, u']^T$ can be written as

$$\begin{aligned}
\dot{\theta} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \end{bmatrix} &= \begin{bmatrix} \alpha_1(\theta_1) \\ \alpha_2(\theta_2) \\ \alpha_3(\theta_3) \end{bmatrix} = \begin{bmatrix} T_1\Phi_1T_1^{-1}\theta_1 \\ T_2\Phi_2T_2^{-1}\theta_2 \\ T_3\Phi_3T_3^{-1}\theta_3 \end{bmatrix} \\
g(\mathbf{x},\mathbf{u}) &= \beta(\theta) = \begin{bmatrix} 0_{3\times1} \\ 0 \\ T_1^{-1}\theta_1 \\ T_2^{-1}\theta_2 \\ T_3^{-1}\theta_3 \end{bmatrix}.
\end{aligned}$$

To design the internal model, choose controllable pair (M_i, N_i) with M_i Hurwitz, choose T_i in the steady-state generator, so that

$$T_i\Phi_i - M_iT_i = N_i \tag{2.47}$$

and then let

$$\dot{\eta} = M\eta + Nu' \tag{2.48}$$

which is the candidate internal model with $M = \text{block diag}(M_1, M_2, M_3)$, $N = \text{block diag}(N_1, N_2, N_3)$, and $\Phi = \text{block diag}(\Phi_1, \Phi_2, \Phi_3)$. Check the condition (2.22), it is easy to find

$$M\theta(v, w) + N\mathbf{u}'(v, w) = \alpha(\theta(v, w)),$$

which proves (2.48) is a proper internal model. Attaching it to the plant (2.45) yields the augmented system.

Performing on the augmented system the following coordinate and input transformations: $\bar{\eta} = \eta - \theta - NJz$, $\bar{u}' = u' - \beta_u(\eta)$ with $\beta_u(\eta) = T^{-1}\eta$, $\bar{e}_v = e_v$, $\bar{e}_4 = \bar{e}_4$ and $\bar{z} = z$, gives

$$\begin{aligned}
\dot{\bar{\eta}} &= M\eta + Nu' - T\Phi T^{-1}\theta - NJ\dot{z} \\
&= M\bar{\eta} + M\theta + MNJz + Nu' - T\Phi T^{-1}\theta - Nu' - Nd - NF(z, e_v, v, w) \\
&= M\bar{\eta} + MNJz - NF(z, e_v, v, w), && (2.49) \\
J\dot{\bar{z}} &= J\dot{z} = T^{-1}\theta + u' + F(z, e_v, v, w) \\
&= T^{-1}\bar{\eta} + T^{-1}NJz + \bar{u}' + F(z, e_v, v, w), && (2.50) \\
\dot{\bar{e}}_v &= \dot{e}_v = 0.5\left[(1 + \bar{e}_4)I_3 + e_v^\times\right](z - Ke_v), \\
\dot{\bar{e}}_4 &= -0.5e_v^T(z - Ke_v), && (2.51)
\end{aligned}$$

where

$$\begin{aligned}
F(z, e_v, v, w) &= -\left[(z - Ke_v + C\Omega_d)^\times J_w(z - Ke_v + C\Omega_d) - \Omega_d^\times J_w\Omega_d\right] \\
&+ J_w\left[(z - Ke_v)^\times C\Omega_d - C\dot{\Omega}_d + \dot{\Omega}_d\right] + 0.5J_wK\left[(1 + \bar{e}_4(e_v))\,I_3 + e_v^\times\right](z - Ke_v).
\end{aligned}$$

By now, we have converted the global robust output regulation problem of system (2.45) to the global robust stabilization problem of the new system (2.49), (2.50) and (2.51). For the rest, we first design the controller $\bar{u}'$, so that the new system is asymptotically stable.

Notice that $F(0, 0, v, w) = 0$ for any v and w since $C = I_3$ when $e_v = 0$; $F(z, e_v, v, w)$ is C^1 for both z and e_v; and J_w, v and e_v are bounded. The following inequality stands:

$$\|F(z, e_v, v, w)\| \leq \|z\|f_1(z) + \|e_v\|f_2(e_v) \leq \|z\|f_1(z) + D\|e_v\|, \tag{2.52}$$

where f_1 and f_2 are some smooth positive nondecreasing functions and D is some positive constant. In fact,

$$\begin{aligned}
\|F(z,e_v,v,w)\| &\leq \|(z-Ke_v+C\Omega_d)^\times J_w(z-Ke_v+C\Omega_d)-\Omega_d^\times J_w\Omega_d\| \\
&\quad +\|J_w\left[(z-Ke_v)^\times C\Omega_d - C\dot{\Omega}_d+\dot{\Omega}_d\right]\| + \|0.5JK\left[(1+\bar{e}_4)I_3+e_v^\times\right](z-Ke_v)\| \\
&\leq \|\Omega_d^\times J_w\Omega_d-(C\Omega_d)^\times wC\Omega_d\| + \|(z-Ke_v)^\times J_w(z-Ke_v)\| \\
&\quad +\|(z-Ke_v)^\times J_wC\Omega_d\| + \|(C\Omega_d)^\times J_w(z-Ke_v)\| \\
&\quad +\|J_w\|\left[\|(z-Ke_v)^\times C\Omega_d\| + \|-C+I_3\|\|\dot{\Omega}_d\|\right] + \|JK\|\|z-Ke_v\| \\
&\leq \|\Omega_d^\times J_w\Omega_d-(C\Omega_d)^\times J_wC\Omega_d\| + \|z-Ke_v\|^2\|J_w\| + 2\|z-Ke_v\|\|J_w\|\|\Omega_d\| \\
&\quad +\|J_w\|\left[\|z-Ke_v\|\|\Omega_d\| + \|-C+I_3\|\|\dot{\Omega}_d\|\right] + \|JK\|\|z-Ke_v\| \\
&= \|\Omega_d^\times J_w\Omega_d-(C\Omega_d)^\times J_wC\Omega_d\| + \|JK\|\|z-Ke_v\| \\
&\quad +\|J_w\|\left[3\|z-Ke_v\|\|\Omega_d\| + \|-C+I_3\|\|\dot{\Omega}_d\| + \|z-Ke_v\|^2\right],
\end{aligned} \tag{2.53}$$

here, the inequalities

$$\|e_v\| \leq 1, \ |e_4| \leq 1,$$

$$\|0.5\left[(1+\bar{e}_4)I_3+e_v^\times\right]\| \leq \|0.5(I_3+e_v^\times)\| \leq 1,$$

the last equation of (2.5), and (2.2) are used.

On the other hand,

$$C = I_3 - 2{e_v}^T e_v I_3 + 2e_v{e_v}^T - 2e_4e_v^\times,$$

w, $\dot{\Omega}_d$ and Ω_d are bounded, so the following inequalities hold:

$$\begin{aligned}
\|-C+I_3\| &= \|-2{e_v}^Te_vI_3+2e_v{e_v}^T-2e_4e_v^\times\| \\
&\leq \|-2{e_v}^Te_vI_3\| + \|2e_v{e_v}^T\| + \|-2e_4e_v^\times\| \leq 6\|e_v\|, \\
\|\Omega_d^\times J_w\Omega_d-(C\Omega_d)^\times J_wC\Omega_d\| &= \|\Omega_d^\times J_w\Omega_d - \left[(C-I_3+I_3)\Omega_d\right]^\times J_w(C-I_3+I_3)\Omega_d\| \\
&= \|\Omega_d^\times J_w\Omega_d - \left[(C-I_3)\Omega_d\right]^\times J_wC\Omega_d - \Omega_d^\times J_w(C-I_3+I_3)\Omega_d\| \\
&= \|-\left[(C-I_3)\Omega_d\right]^\times J_wC\Omega_d - \Omega_d^\times J_w(C-I_3)\Omega_d\| \\
&\leq \|-\left[(C-I_3)\Omega_d\right]^\times J_wC\Omega_d\| + \|-\Omega_d^\times J_w(C-I_3)\Omega_d\| \\
&\leq \|C-I_3\|\|\Omega_d\|^2\|J_w\|\|C\| + \|\Omega_d\|^2\|J_w\|\|C-I_3\| \\
&\leq 12\|e_v\|\|\Omega_d\|^2\|J_w\|,
\end{aligned}$$

$\|J_w\| \leq L_1$, $\|\dot{\Omega}_d\| \leq L_2$, $\|\Omega_d\| \leq L_3$, $\|J\| \leq L_4$ and $\|K\| \leq k_2$, where L_1, L_2, L_3, L_4 and k_2 are known positive constants. In this way, from (2.53) we have

$$\begin{aligned}
\|F(z,e_v,v,w)\| &\leq (12L_3^2L_1+6L_1L_2)\|e_v\|+L_1\|z-Ke_v\|^2+(3L_1L_3+L_4k_2)\|z-Ke_v\| \\
&\leq (12L_3^2L_1+6L_1L_2)\|e_v\|+L_1(\|z\|+k_2\|e_v\|)^2+(3L_1L_3+L_4k_2)(\|z\|+k_2\|e_v\|) \\
&= (12L_3^2L_1+6L_1L_2+3L_1L_3k_2+L_4k_2^2)\|e_v\|+L_1\|z\|^2+2L_1k_2\|z\|\|e_v\| \\
&\quad +L_1k_2^2\|e_v\|^2+(3L_1L_3+L_4k_2)\|z\| \\
&\leq (L_1k_2^2+3L_1L_3k_2+12L_3^2L_1+6L_1L_2+L_4k_2^2)\|e_v\| \\
&\quad +(L_1\|z\|+2L_1k_2+3L_1L_3+L_4k_2)\|z\|. \qquad (2.54)
\end{aligned}$$

Choosing

$$f_1(\|z\|)=L_1\|z\|+2L_1k_2+3L_1L_3+L_4k_2$$

and

$$f_2(\|e_v\|)=D=L_1k_2^2+3L_1L_3k_2+12L_3^2L_1+6L_1L_2+L_4k_2^2,$$

the inequality (2.52) stands.

For (2.49), choose candidate Lyapunov function $V_1=0.5\bar{\eta}^T\bar{\eta}$ and the derivative of V_1 can be written as follows:

$$\begin{aligned}
\dot{V}_1 &= \bar{\eta}^TM\bar{\eta}+\bar{\eta}^TMNJz-\bar{\eta}^TNF(z,e_v,v,w) \\
&\leq -k_3\|\bar{\eta}\|^2+\|\bar{\eta}\|\|MNJ\|\|z\|+\|\bar{\eta}\|\|N\|\left[f_1(z)\|z\|+D\|e_v\|\right] \\
&\leq -k_3\|\bar{\eta}\|^2+\|\bar{\eta}\|\left[\|MNJ\|+f_1(z)\|N\|\right]\|z\|+D\|N\|\|\bar{\eta}\|\|e_v\| \\
&\leq -k_3\|\bar{\eta}\|^2+0.5\epsilon_1^2\|MNJ\|\|\bar{\eta}\|^2+\frac{0.5}{\epsilon_1^2}\|MNJ\|\|z\|^2 \\
&\quad +0.5\|N\|\|\bar{\eta}\|^2+0.5\|N\|\left[f_1^2(\|z\|)\|z\|^2\right]+\frac{0.5}{\epsilon_2^2}D\|N\|\|\bar{\eta}\|^2+0.5\epsilon_2^2D\|N\|\|e_v\|^2 \\
&= (-k_3+0.5\epsilon_1^2\|MNJ\|+0.5\|N\|+\frac{0.5}{\epsilon_2^2}D\|N\|)\|\bar{\eta}\|^2 \\
&\quad +(0.5\|N\|f_1^2(\|z\|)+\frac{0.5}{\epsilon_1^2}\|MNJ\|)\|z\|^2+0.5\epsilon_2^2D\|N\|\|e_v\|^2, \qquad (2.55)
\end{aligned}$$

here $k_3>0$ exists since M is Hurwitz and ϵ_1, ϵ_2 are positive constants and to be chosen later.

For (2.51), choose candidate Lyapunov function as $V_2=e_v^Te_v+\bar{e}_4^2$, and its derivative is given as

$$
\begin{aligned}
\dot{V}_2 &= e_v^T\left((1+\bar{e}_4)I_3 + e_v^\times\right)(z-Ke_v) - \bar{e}_4 e_v^T(z-Ke_v) \\
&= (1+\bar{e}_4)e_v^T(z-Ke_v) + e_v^T e_v^\times(z-Ke_v) - \bar{e}_4 e_v^T(z-Ke_v) \\
&= e_v^T(z-Ke_v) + \bar{e}_4 e_v^T(z-Ke_v) + e_v^T e_v^\times(z-Ke_v) - \bar{e}_4 e_v^T(z-Ke_v) \\
&= e_v^T(z-Ke_v) \\
&\le -k_1\|e_v\|^2 + 0.5\epsilon_3^2\|e_v\|^2 + \frac{0.5}{\epsilon_3^2}\|z\|^2 \\
&= (-k_1+0.5\epsilon_3^2)\|e_v\|^2 + \frac{0.5}{\epsilon_3^2}\|z\|^2, \qquad (2.56)
\end{aligned}
$$

where, $e_v^T e_v^\times = -(e_v^\times e_v)^T = 0$ from the first and third equations of (2.5), $k_1 > 0$ is the minimum eigenvalue of K and ϵ_3 is positive constant and to be chosen later.

For (2.50), let $V_z(z) = 0.5z^T Jz$, and we have

$$
\begin{aligned}
\dot{V}_z &\le z^T T^{-1}\bar{\eta} + z^T\bar{u}' + z^T T^{-1}NJz + z^T F(z,e_v,v,w) \\
&\le \|T^{-1}\|\|z\|\|\bar{\eta}\| + z^T\bar{u}' + \|T^{-1}NJ\|\|z\|^2 + \|z\|\left[f_1(\|z\|)\|z\| + D\|e_v\|\right] \\
&\le 0.5\epsilon_4^2\|T^{-1}\|\|z\|^2 + \frac{0.5}{\epsilon_4^2}\|T^{-1}\|\|\bar{\eta}\|^2 + z^T\bar{u}' + \|T^{-1}NJ\|\|z\|^2 + f_1(\|z\|)\|z\|^2 \\
&\quad +0.5\epsilon_5^2 D\|z\|^2 + \frac{0.5}{\epsilon_5^2}D\|e_v\|^2 \\
&= \left(0.5\epsilon_4^2\|T^{-1}\| + \|T^{-1}NJ\| + f_1(\|z\|) + 0.5\epsilon_5^2 D\right)\|z\|^2 \\
&\quad +\frac{0.5}{\epsilon_4^2}\|T^{-1}\|\|\bar{\eta}\|^2 + z^T\bar{u}' + \frac{0.5}{\epsilon_5^2}D\|e_v\|^2, \qquad (2.57)
\end{aligned}
$$

where ϵ_4 and ϵ_5 are positive constants and to be chosen later.

We design

$$
\bar{u}' = -K_z(\|z\|)z = -\left[k_z + f_1(\|z\|) + 0.5\|N\|f_1^2(\|z\|)\right]I_3 z, \qquad (2.58)
$$

where k_z is positive number and determined later. Choose the overall Lyapunov function $W = V_1 + V_2 + V_z$, then the derivative of W along the trajectory of system (2.49) (2.51) and (2.50) is achieved from (2.55), (2.56) and (2.57) as follows:

$$
\begin{aligned}
\frac{dW(z,e_v,\bar{\eta})}{dt} &\le (0.5\epsilon_4^2\|T^{-1}\| - k_z + \|T^{-1}NJ\| + 0.5\epsilon_5^2 D + \frac{0.5}{\epsilon_3^2} + \frac{0.5}{\epsilon_1^2}\|MNJ\|)\|z\|^2 \\
&+ \left[\frac{0.5}{\epsilon_4^2}\|T^{-1}\| - k_3 + 0.5\epsilon_1^2\|MNJ\| + (0.5+\frac{0.5}{\epsilon_2^2})\|N\|\right]\|\bar{\eta}\|^2 \\
&+ (\frac{0.5D}{\epsilon_5^2} - k_1 + 0.5\epsilon_3^2 + 0.5\epsilon_2^2 D^2\|N\|)\|e_v\|^2.
\end{aligned}
$$

Because of the fact that the pair (M,N) is controllable implies the pair(kM,N) is also controllable with $k \in R$ and $k \neq 0$, and the fact that J has a known bound, choose appropriately k, k_3,

k_1, ϵ_1, ϵ_2, ϵ_3, ϵ_4 and ϵ_5 as well as modify M and T, so that

$$\left[\frac{0.5}{\epsilon_4^2}\|T^{-1}\| - k_3 + 0.5\epsilon_1^2\|MNJ\| + (0.5 + \frac{0.5}{\epsilon_2^2})\|N\|\right] \leq -\epsilon_\eta \tag{2.59}$$

and

$$(\frac{0.5D}{\epsilon_5^2} - k_1 + 0.5\epsilon_3^2 + 0.5\epsilon_2^2 D^2\|N\|) \leq -\epsilon_{e_v}, \tag{2.60}$$

here, ϵ_η and ϵ_{e_v} are positive constants. In fact, noticing that D is related to K and then k_1, we can first choose $k_1 > 0$ freely and fix it. Choose $\epsilon_{e_v} < k_1$ and choose ϵ_2, ϵ_3 and ϵ_5 so that

$$\begin{aligned}\frac{0.5D}{\epsilon_5^2} &\leq \frac{1}{3}(k_1 - \epsilon_{e_v}),\\ 0.5\epsilon_3^2 &\leq \frac{1}{3}(k_1 - \epsilon_{e_v}),\\ 0.5\epsilon_2^2 D^2\|N\| &\leq \frac{1}{3}(k_1 - \epsilon_{e_v}).\end{aligned}$$

Then (2.60) stands. Since ϵ_2 is fixed, we choose k_3 so that $k_3 > (0.5 + \frac{0.5}{\epsilon_2^2})\|N\|$ and choose ϵ_η so that $\epsilon_\eta < k_3 - (0.5 + \frac{0.5}{\epsilon_2^2})\|N\|$. Then, we need to modify M. Choose k so that $\bar{\eta}^T(kM)\bar{\eta} \leq -k_3\|\bar{\eta}\|^2$ for any $\bar{\eta}$, then look at kM as the new M. Accordingly, we calculate T again with equation (2.47). Choose ϵ_1, and ϵ_4 so that

$$\begin{aligned}\frac{0.5}{\epsilon_4^2}\|T^{-1}\| &\leq 0.5\left[k_3 - \epsilon_\eta - (0.5 + \frac{0.5}{\epsilon_2^2})\|N\|\right]\\ 0.5\epsilon_1^2\|MNJ\| &\leq 0.5\left[k_3 - \epsilon_\eta - (0.5 + \frac{0.5}{\epsilon_2^2})\|N\|\right].\end{aligned}$$

Then (2.59) stands. In the following, choose positive constant ϵ_z freely and choose k_z so that

$$\left[0.5\varepsilon_1^2\|T^{-1}\| - k_z + \|T^{-1}NJ\| + 0.5\varepsilon_2^2 D + \frac{0.5}{\epsilon_3^2} + \frac{0.5}{\epsilon_1^2}\|MNJ\|\right] \leq -\epsilon_z.$$

In this way,

$$\frac{dW(z, e_v, \bar{\eta})}{dt} \leq -\epsilon_\eta\|\bar{\eta}\|^2 - \epsilon_{e_v}\|e_v\|^2 - \epsilon_z\|z\|^2 \leq 0.$$

By LaSalle Theorem, the equilibrium point at the origin of the close-loop system (2.49), (2.50), (2.51), and (2.58) is globally asymptotic stable.

Here we do not require the bound of J be arbitrarily small, because, for any large but bounded J, we can always find appropriate L_1 as well as L_4, and hence we can design proper k_1, M and k_z so that the inequalities above stand.

Overall, the controller is given as follows:

$$\begin{aligned}u &= -K_z(\|\omega + Ke_v\|)(\omega + Ke_v) + T^{-1}\eta + (\omega + C\Omega_d)^\times J_0(\omega + C\Omega_d)\\ &- J_0(\omega^\times C\Omega_d - C\dot{\Omega}_d) - J_0 K\left[0.5(e_4 I_3 + e_v^\times)\omega\right]\\ \dot{\eta} &= M\eta - NK_z(\|\omega + Ke_v\|)(\omega + Ke_v) + NT^{-1}\eta,\end{aligned}$$

which solves the global output regulation problem for the composite system (2.9), (2.10), (2.33) and (2.44).

2.5 Simulation

2.5.1 Case I: without unknown parameters

A control problem of a rigid-body micro-satellite is simulated to demonstrate the performance of control law achieved ahead. The spacecraft is assumed to have the inertia matrix [1] of

$$J = \begin{pmatrix} 20 & 1.2 & 0.9 \\ 1.2 & 17 & 1.4 \\ 0.9 & 1.4 & 15 \end{pmatrix} kg \cdot m^2$$

which is known to the controller. The target angular velocity Ω_d of frame F_r to be tracked is supposed to be expressed in the body frame F_r as

$$\Omega_d = \begin{pmatrix} 5\sin(1\pi t/100) \\ 5\sin(2\pi t/100) \\ 5\sin(3\pi t/100) \end{pmatrix} rad/s$$

and the initial target unit quaternion

$$q_c(0) = [0, 0, 0, 1]^T.$$

For the numerical simulation of controller, we assume that the initial attitude orientation of the spacecraft in the frame F_b is

$$q(0) = [0.3, -0.2, -0.3, 0.8832]^T,$$

and the initial value of the angular velocity in in the frame F_b is

$$\Omega(0) = [0.06, -0.04, 0.05]^T rad/s.$$

The gains of the control law are $K = 0.2I_3$, $K_z = 500I_3$.

We consider the disturbance as follows:

$$d(t) = \begin{pmatrix} 2\sin(\pi t/5) + \sin(2\pi t/5) - \cos(3\pi t/5) \\ -2\sin(\pi t/5) - 2\sin(2\pi t/5) + \cos(3\pi t/5) \\ 2\sin(\pi t/5) + \cos(2\pi t/5) + \sin(3\pi t/5) \end{pmatrix} Nm,$$

and

$$\begin{aligned} d_1 &= -2v_{d11} - v_{d21} + v_{d32}, \\ d_2 &= 2v_{d11} + v_{d21} - v_{d32}, \\ d_3 &= -2v_{d11} + v_{d22} - v_{d31} \end{aligned}$$

with

$$\begin{pmatrix} \dot{v}_{d11} \\ \dot{v}_{d12} \end{pmatrix} = \begin{pmatrix} 0 & \pi/5 \\ -\pi/5 & 0 \end{pmatrix} \begin{pmatrix} v_{d11} \\ v_{d12} \end{pmatrix}$$

$$\begin{pmatrix} \dot{v}_{d21} \\ \dot{v}_{d22} \end{pmatrix} = \begin{pmatrix} 0 & 2\pi/5 \\ -2\pi/5 & 0 \end{pmatrix} \begin{pmatrix} v_{d21} \\ v_{d22} \end{pmatrix}$$

$$\begin{pmatrix} \dot{v}_{d31} \\ \dot{v}_{d32} \end{pmatrix} = \begin{pmatrix} 0 & 3\pi/5 \\ -3\pi/5 & 0 \end{pmatrix} \begin{pmatrix} v_{d31} \\ v_{d32} \end{pmatrix}.$$

It is easy to see that

$$\Phi_1 = \Phi_2 = \Phi_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -2.215 & 0 & -7.6369 & 0 & -5.527 & 0 \end{pmatrix}.$$

And choose

$$M_1 = M_2 = M_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -3 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -3 & -1 \end{pmatrix}$$

and

$$N_1 = N_2 = N_3 = (0\ 1\ 0\ 1\ 0\ 1)^T.$$

Figure 2.2. shows that the controller we design with output regulation method achieves the attitude tracking and disturbance rejection.

2.5.2 Case II: with unknown parameters

In order to simplify the calculation, we assume

$$J_w = J_{w_1} = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} kg \cdot m^2, \quad or \quad J_w = J_{w_2} = \begin{pmatrix} -30 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} kg \cdot m^2,$$

$J, \Omega_d, q_c(0), q(0), \Omega(0)$ and K are the same with case I.

In this case,

$$\begin{aligned} \boldsymbol{\Omega}_{d1} &= w_1 v_{\Omega_{12}} a_1, \\ \boldsymbol{\Omega}_{d2} &= w_1 v_{\Omega_{31}} v_{\Omega_{11}}, \\ \boldsymbol{\Omega}_{d3} &= -w_1 v_{\Omega_{21}} v_{\Omega_{11}}. \end{aligned}$$

where

$$\begin{aligned} A_{\Omega 1} &= \begin{bmatrix} 0 & \pi/100 \\ -\pi/100 & 0 \end{bmatrix} \\ A_{\Omega 2} &= \begin{bmatrix} 0 & 2\pi/100 \\ -2\pi/100 & 0 \end{bmatrix} \\ A_{\Omega 3} &= \begin{bmatrix} 0 & 3\pi/100 \\ -3\pi/100 & 0 \end{bmatrix}. \end{aligned}$$

And then

$$\Phi_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -0.0022 & 0 & -2.2226 & 0 & -7.6423 & 0 & -5.528 & 0 \end{pmatrix}$$

$$\Phi_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -0.0001 & 0 & -0.0442 & 0 & -2.3661 & 0 & -7.746 & 0 & -5.5467 & 0 \end{pmatrix}$$

$$\Phi_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1.9419 \times 10^{-5} & 0 & -0.0219 & 0 & -2.2905 & 0 & -7.6914 & 0 & -5.5368 & 0 \end{pmatrix}$$

Choose

$$M_1 = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -3 & -1 \end{pmatrix},$$

$$M_2 = M_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -24 & -50 & -35 & -10 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -3 & -1 \end{pmatrix},$$

and

$$N_1 = (0\,1\,0\,1\,0\,1\,0\,1)^T,$$

$$N_2 = N_3 = (0\,0\,0\,1\,0\,1\,0\,1\,0\,1)^T.$$

To design the controller, we choose $L_1 = 30$, $L_3 = 10$, $L_4 = 30$, $k_2 = 0.2$ and $k_z = 500$.

Figure 2.3 shows the system behavior with uncertainty $J_w = J_{w_1}$ and under the controller designed in case I. Figure 2.4 shows the system behavior under the controller we design in case II without system uncertainty. Figure 2.5 shows the system behavior under the controller designed in case II and with system uncertainty $J_w = J_{w_1}$. Figure 2.6 shows the system behavior under the controller designed in case II and with system uncertainty $J_w = J_{w_2}$. From the figures, as expected, the parameter variations do not affect the tracking response of states with the controller designed in case II, which is in contrast with the controller designed in case I, where the same parameter variations significantly affect the steady state response of the states.

2.6 Conclusions

In this chapter, the attitude tracking control problem is addressed for a nonlinear system with time-varying disturbances and parameter uncertainties. The general framework for output regulation problem is used to obtain the attitude tracking and disturbance rejection of spacecraft motion. The advantages of this framework not only lie in its disturbance rejection but also in overcoming the presence of uncertain parameter. The numerical simulation shows the good performance of the method. Compared to the work in [1] to [6], the controller here achieves the asymptotic disturbance rejection instead of disturbance attenuation or zero-disturbance.

However, the design presented here does not consider the degree of optimality of the controller. Besides it is also interesting to further investigate the problem of designing the controller to improve the transient response and minimize energy cost. In [11], the author also pointed out the assumption of three independent inputs is not directly applicable if the spacecraft is equipped with magnetic coils as attitude actuators. Then the case of only two independent inputs available will be under consideration.

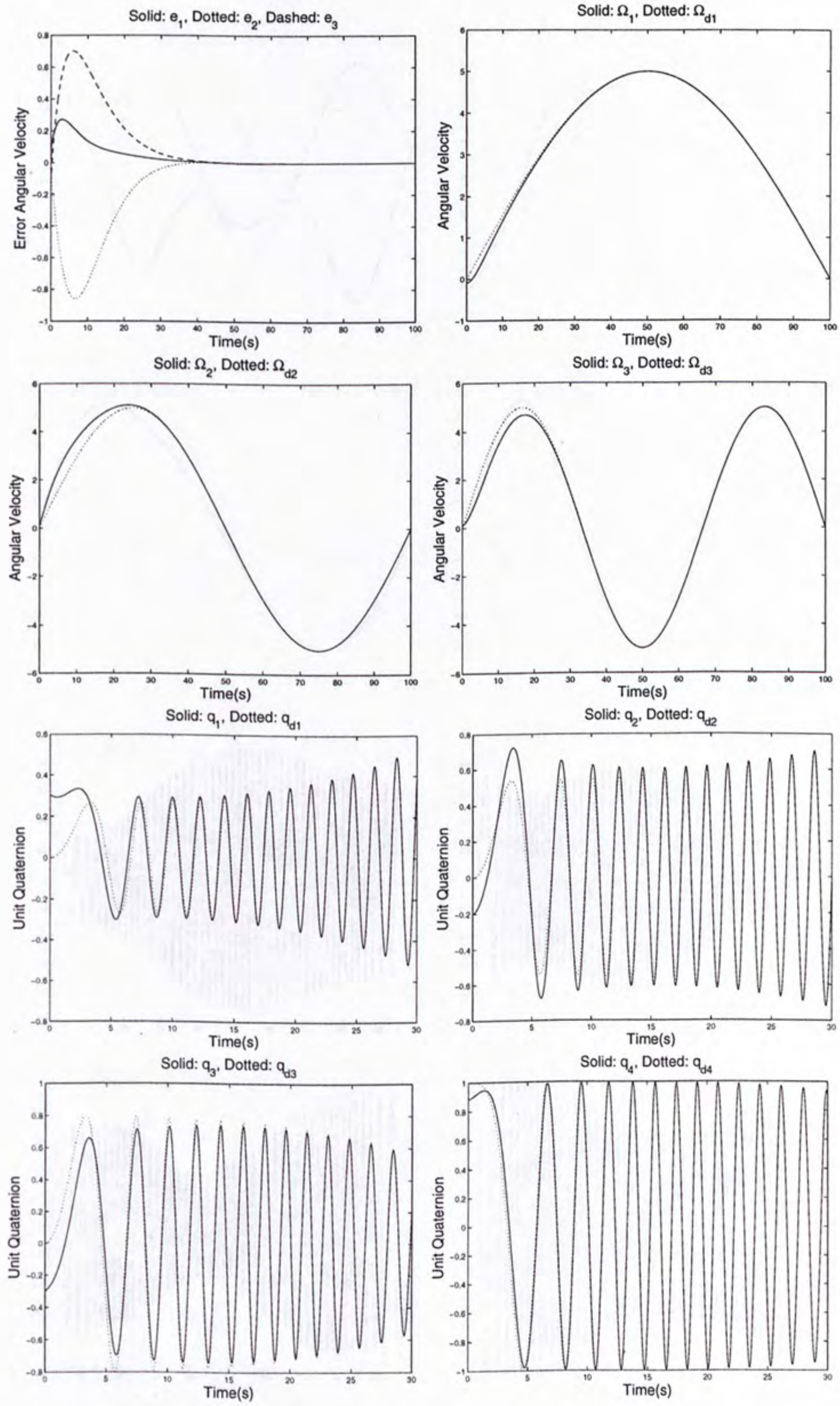

Figure 2.2: The first figure is the error angular velocities between Ω and Ω_d without system uncertainty, and the rest figures are comparisons of reference states and practical states.

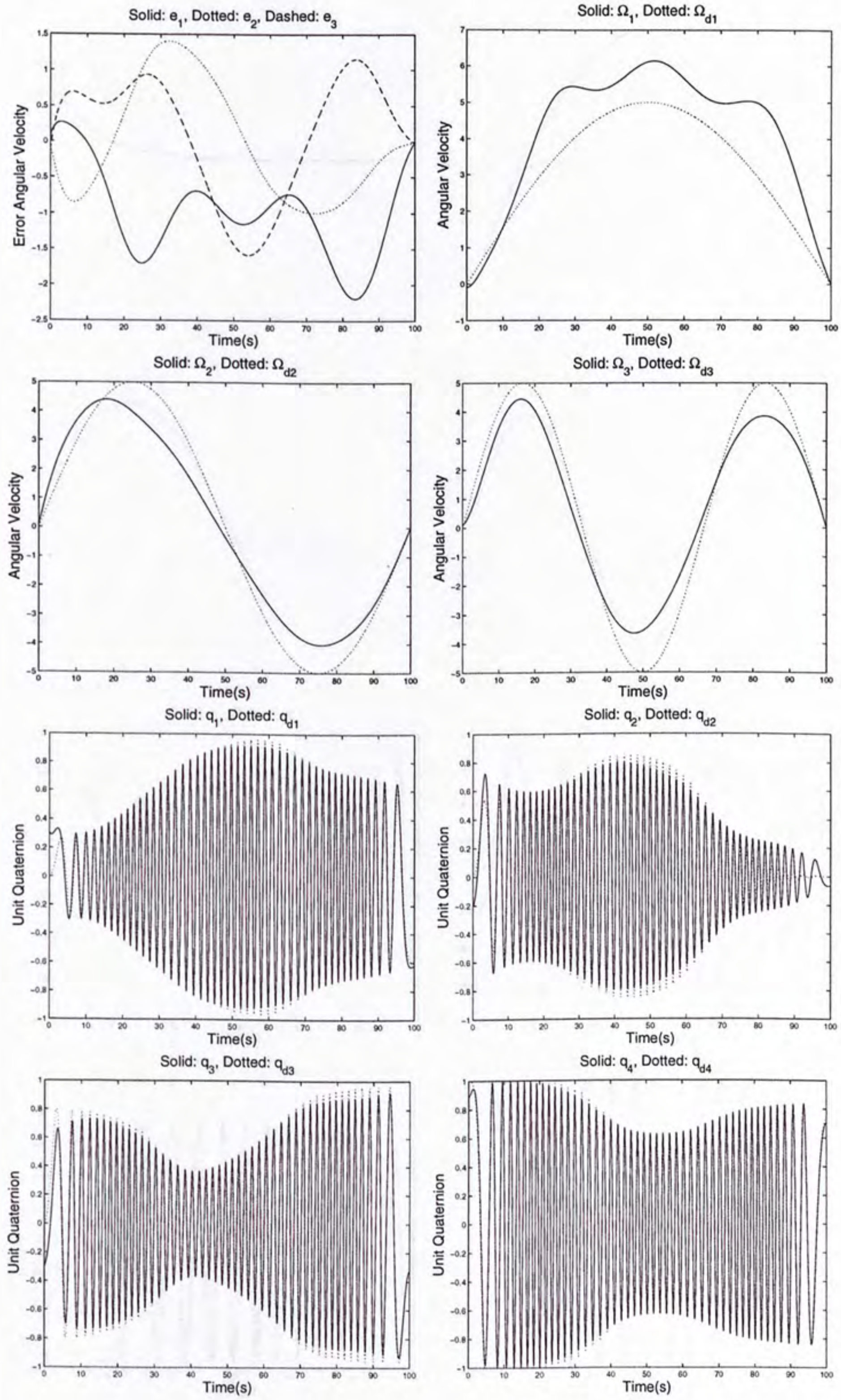

Figure 2.3: The first figure is the error angular velocities between Ω and Ω_d with system uncertainty $J_w = J_{w_1}$, and the rest figures are comparisons of reference states and practical states.

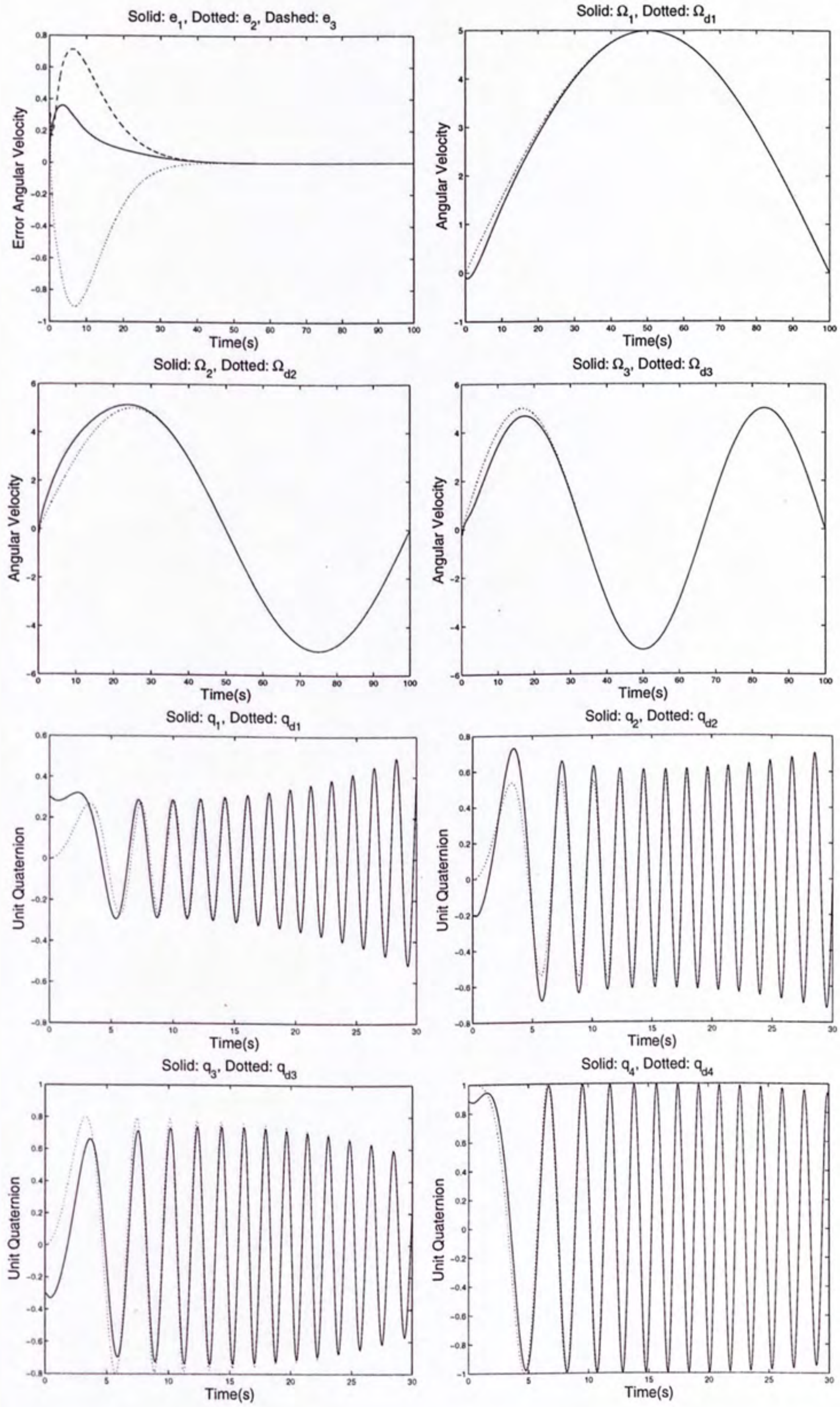

Figure 2.4: The first figure is the error angular velocities between Ω and Ω_d without system uncertainty, and the rest figures are comparisons of reference states and practical states.

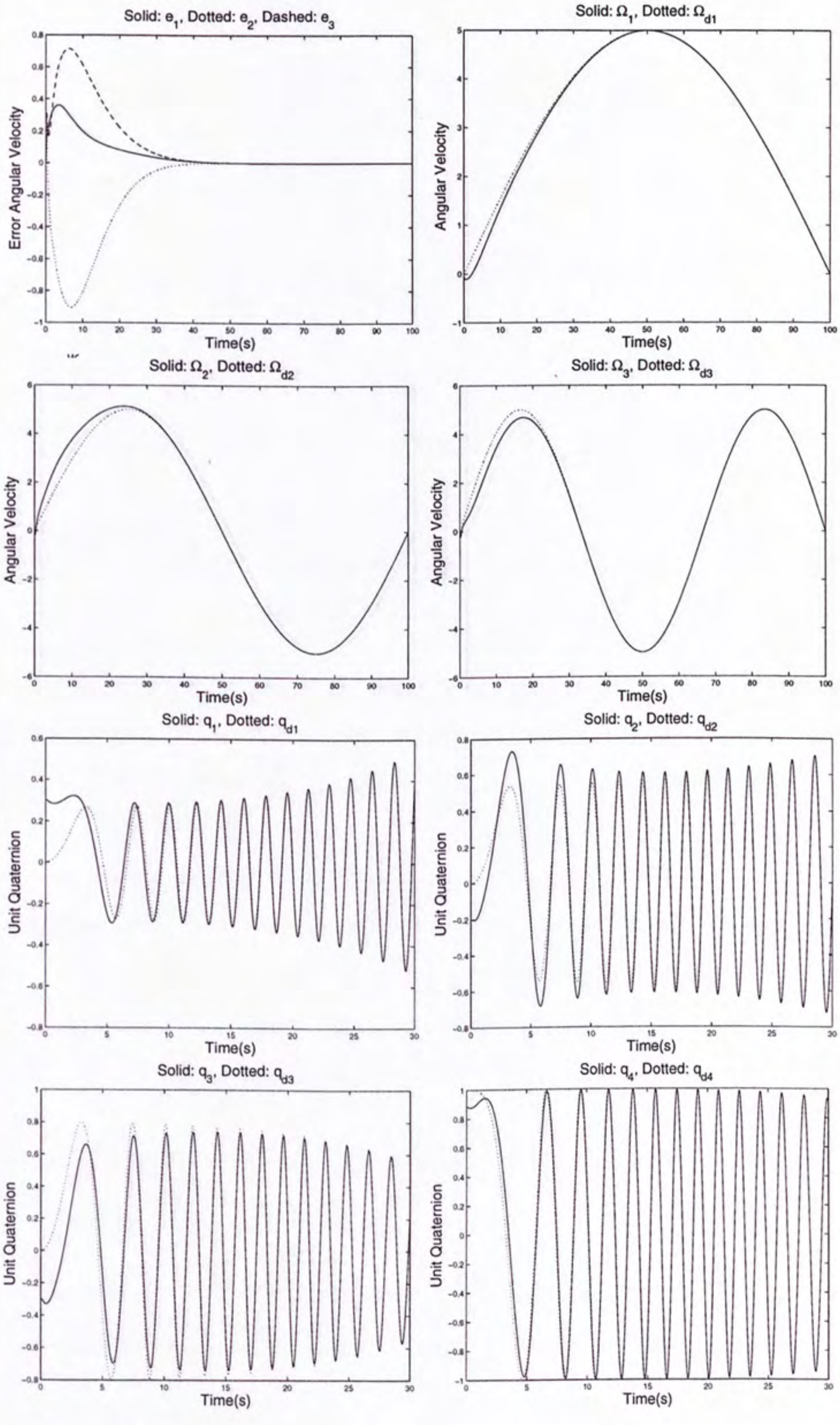

Figure 2.5: The first figure is the error angular velocities between Ω and Ω_d with system uncertainty $J_w = J_{w1}$, and the rest figures are comparisons of reference states and practical states.

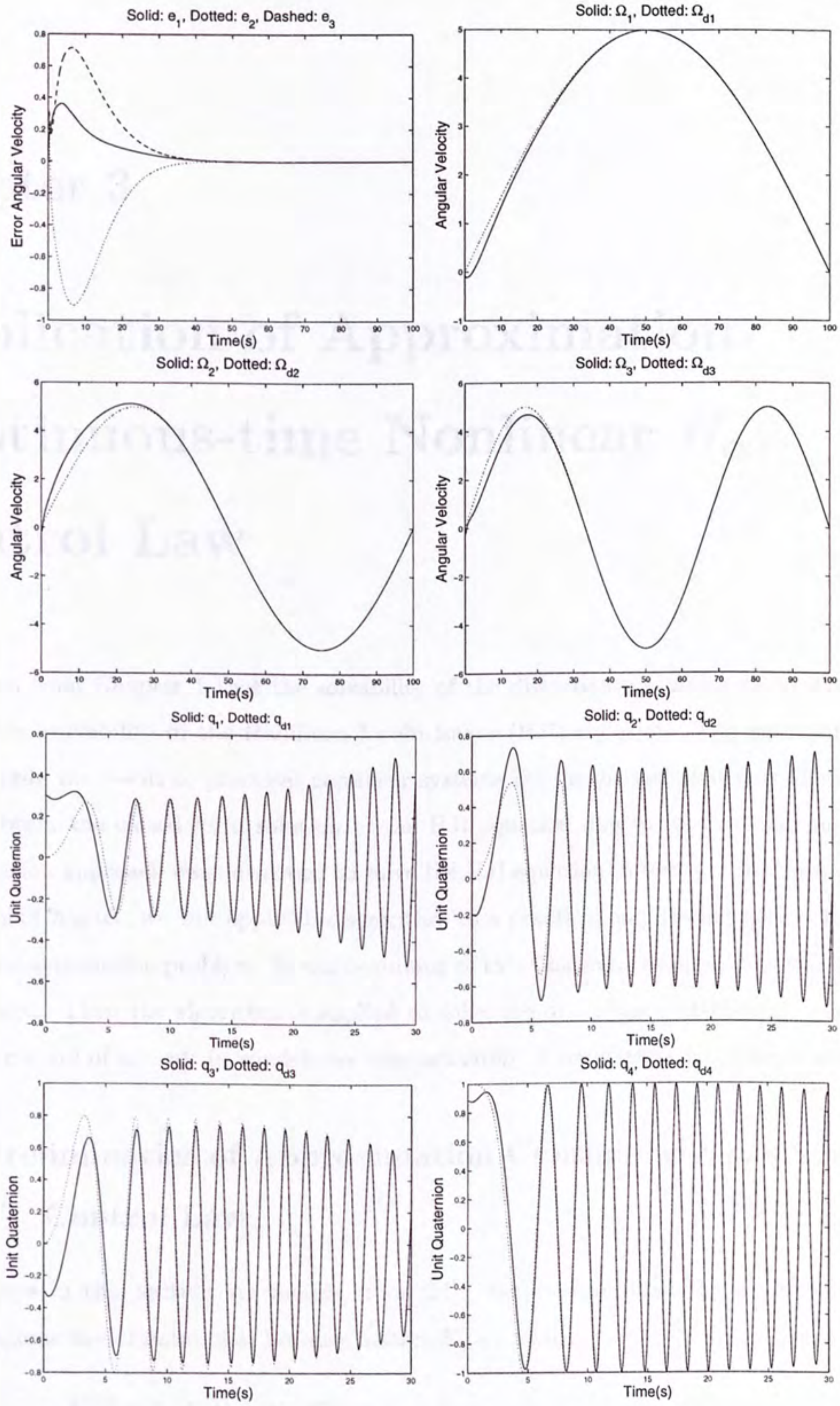

Figure 2.6: The first figure is the error angular velocities between Ω and Ω_d with system uncertainty $J_w = J_{w_2}$, and the rest figures are comparisons of reference states and practical states.

Chapter 3

Application of Approximation Continuous-time Nonlinear H_∞ Control Law

It is known from Chapter 1 that the solvability of the disturbance attenuation problem comes down to the solvability of the Hamilton-Jacobi-Isaacs (HJI) equation. The main difficulty to directly apply the result to practical nonlinear systems lies on the fact that it is almost impossible to obtain the closed form solution of the HJI equation due to its nonlinear nature. An approximation approach was developed to solve the HJI equation in terms of the Taylor series in [24]. In this Chapter, we will apply this algorithm to a practical engineering system to solve its disturbance attenuation problem. In the beginning of this Chapter, the approximation algorithm is introduced. Then the algorithm is applied to solve the disturbance attenuation problem for the flight control of aircraft in windshears approximately. Finally, the conclusion is given.

3.1 Preliminaries of Approximation Continuous-time Nonlinear H_∞ Control Law

The writings in this section are mainly from [24]. Before describing the algorithm, we first introduce some useful notations. For any matrix K, we define

$$K^{(0)} = 1,\ K^{(1)} = K,\ K^{(i)} = \underbrace{K \otimes K \otimes \cdots \otimes K}_{i\ factors},\ \ i = 2,\ 3,\ \cdots$$

where $\otimes$ stands for the Kronecker product. Also for any n-dimensional vector $x = [x_1, \cdots, x_n]^T$, we define

$$\begin{aligned} x^{[0]} &= 1, \quad x^{[1]} = x \\ x^{[k]} &= [x_1^k \; x_1^{k-1}x_2 \; \cdots \; x_1^{k-1}x_n \; x_1^{k-2}x_2^2 \; x_1^{k-2}x_2x_3 \; \cdots \\ & \quad\; x_1^{k-2}x_2x_n \; \cdots \; x_1^{k-2}x_2x_n \; \cdots \; x_n^k]^T \qquad k \geq 1. \end{aligned}$$

There exist constant matrices M_k and N_k such that

$$x^{[k]} = M_k x^{(k)}, \;\; x^{(k)} = N_k x^{[k]}.$$

For example, with $n = 2$, $x^{(2)}$ and $x^{[2]}$ are given by, respectively,

$$x^{(2)} = \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2x_1 \\ x_2^2 \end{bmatrix}, \quad x^{[2]} = \begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{bmatrix}.$$

Correspondingly, M_2 and N_2 are given by, respectively,

$$M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Obviously, M_k is not exclusive but N_k is. For instance,

$$M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

also satisfies the definition. It is noted that the dimensions of M_k and N_k are $C(n, k)$ by n^k and, respectively, n^k by $C(n, k)$ where

$$C(n,k) \stackrel{def}{=} C_{n+k-1}^k = \frac{\prod_{i=1}^k (n+k-i)}{k!}$$

Also, $M_k N_k = I_n^{[k]}$ where $I_n^{[k]}$ is an identity matrix of dimension $C(n, k)$ [25].

We also list some useful identities [24] involving the Kronecker product as follows:

Lemma 3.1

(i) For $k \geq 1$,

$$\frac{\partial x^{(k)}}{\partial x} = \sum_{i=1}^{k} x^{(i-1)} \otimes I_n \otimes x^{(k-i)} \tag{3.1}$$

where I_n denotes an n by n identity matrix.

(ii) For $x \in R^n$, $y \in R^m$ and $A \in R^{n\times m}$,

$$x^T Ay = row(A)(x \otimes y) \tag{3.2}$$

where $row : R^{n\times m} \to R^{1\times nm}$ is an operator that maps n by m matrix $A = [a_{ij}]$ to a 1 by mn row vector $row(A)$ in the following way:

$$row(A) = [a_{11}\ a_{12}\ \cdots\ a_{1m} \cdots a_{n1} \cdots a_{nm}]$$

(iii) For any integers $i, j, k \geq 0$, and matrix T of dimension n by n^k

$$(x^{(i)} \otimes I_n \otimes x^{(j)})Tx^{(k)} = (I_n^{(i)} \otimes T \otimes I_n^{(j)})x^{(i+j+k)} \tag{3.3}$$

These notations are essential to the algorithm, because the application of these notations together with the lemma provides the possibility that the coefficients of Taylor series solution of HJI equation are governed by a sequence of linear algebraic equations and a algebraic Ricatti equation.

First, using the above notations gives the following unique expression for $V(x)$:

$$V(x) \;=\; \frac{1}{2}x^T Px + \sum_{k=3}^{\infty} P_k x^{[k]}. \tag{3.4}$$

Similarly, the Taylor series representation of $f(x)$ and $h(x)$ can be given by

$$f(x) = \sum_{m=1}^{\infty} A_m x^{(m)}, \quad h_1(x) \;=\; \sum_{m=1}^{\infty} C_m x^{(m)}.$$

On the other hand we can write

$$g_1(x) = B_1 + g_1^{[1+]}(x), \quad g_2(x) = B_2 + g_2^{[1+]}(x),$$

where

$$\begin{aligned} B_1 &= [B_{11}\ \cdots\ B_{1m_1}]\,, \quad B_2 = [B_{21}\ \cdots\ B_{2m_2}] \\ g_1^{[1+]}(x) &= [g_{11}(x)\ \cdots\ g_{1m_1}(x)]\,, \quad g_2^{[1+]}(x) = [g_{21}(x)\ \cdots\ g_{2m_2}(x)] \end{aligned}$$

and write the Taylor series expansions of $g_{1l}(x)$, $l = 1, 2, \cdots, m_1$ and $g_{2l}(x)$, $l = 1, 2, \cdots, m_2$, as follows:

$$g_{1l}(x) \;=\; \sum_{m=1}^{\infty} B_{1l}^m x^{(m)}\,, \quad g_{2l}(x) = \sum_{m=1}^{\infty} B_{2l}^m x^{(m)}.$$

Then we introduce the main results of the algorithm.

Theorem 3.1 The coefficient vector P of the Taylor series expansions of $V(x)$ satisfies the following equation:

$$\frac{1}{2}x^T H_{xx} x + x^T P H_{px} x + \frac{1}{2} x^T P H_{pp} P x = 0, \tag{3.5}$$

where,

$$\begin{aligned} H_{px} &= A, \quad H_{xx} = C_1^T C_1 \\ H_{pp} &= \frac{B_1 B_1^T}{\gamma^2} - B_2 R_2^{-1} B_2^T, \end{aligned}$$

and the coefficient vectors P_k of (3.4) satisfies the following equation

$$P_k = H_k U_k^{-1} \tag{3.6}$$

where

$$U_k = M_k \left(\sum_{i=1}^{k} I_n^{(i-1)} \otimes (H_{px} + H_{pp} P) \otimes I_n^{(k-i)} \right) N_k \tag{3.7}$$

and H_k depends only on $P, P_3, \cdots, P_{k-1}$,

$$\begin{aligned} H_3 &= -\left(E_3 + \frac{F_3 - 2I_3^2}{2} + \frac{I_3^1}{\gamma^2} \right) N_3 \\ H_k &= -\frac{1}{2}\left(Z_k + 2E_k + F_k + \frac{2I_k^1 + G_k^1}{\gamma^2} - 2I_k^2 - G_k^2 \right) N_k, \quad k = 4, 5, \cdots . \end{aligned} \tag{3.8}$$

Lemma 3.2 The eigenvalues of U_k are given by

$$\lambda = l_1 \lambda_1 + \cdots + l_n \lambda_n \tag{3.9}$$

where $l_1 + \cdots + l_n = k$, $l_1, \cdots, l_n = 0, 1, \cdots, k$ and $\lambda_1, \cdots, \lambda_n$ are eigenvalues of $H_{px} + H_{pp} P$.

The above lemma immediately leads to the existence condition of the Taylor series solution of (3.4) as follows.

Corollary 3.1 There exists a unique solution P_k of (3.6) for $k \geq 3$ if and only if for all $k \geq 3$, all $l_1, l_2, \cdots, l_n \in \{0, 1, 2, \cdots\}$ such that $l_1 + l_2 + \cdots + l_n = k$

$$l_1 \lambda_1 + \cdots + l_n \lambda_n \neq 0 \tag{3.10}$$

In the following, we introduce the parameters appeared in (3.7) and (3.8),

$$Z_k = \sum_{l=3}^{k-1} row(S_l H_{pp} S_{k-l+2}^T), \quad k = 4, 5, \cdots \tag{3.11}$$

$$E_k = \sum_{l=2}^{k-1} row(S_l A_{k-l+1}), \quad k = 3, 4, \cdots, \tag{3.12}$$

$$F_k = \sum_{l=1}^{k-1} row(C_l^T C_{k-l}), \;\; k = 2, 3, \cdots, \tag{3.13}$$

$$Y_{ij}^k = B_{ij}^T S_{k+1}^T, \;\; k = 1, 2, \cdots, \tag{3.14}$$

$$W_{ij}^k = \sum_{l=2}^{k} row(S_l B_{ij}^{k+1-l}), \;\; k = 2, 3, \cdots, \tag{3.15}$$

$$I_k^1 = \sum_{l=2}^{k-1} \sum_{j=1}^{m_1} row((W_{1j}^l)^T Y_{1j}^{k-l}), k = 3, 4, \cdots, \tag{3.16}$$

$$I_k^2 = \sum_{l=2}^{k-1} \sum_{j=1}^{m_2} row((W_{2j}^l)^T R_2^{-1} Y_{2j}^{k-l}), k = 3, 4, \cdots, \tag{3.17}$$

$$G_k^1 = \sum_{l=2}^{k-2} \sum_{j=1}^{m_1} row((W_{1j}^l)^T W_{1j}^{k-l}), k = 4, 5, \cdots, \tag{3.18}$$

$$G_k^2 = \sum_{l=2}^{k-2} \sum_{j=1}^{m_1} row((W_{2j}^l)^T R_2^{-1} W_{2j}^{k-l}), k = 4, 5, \cdots, \tag{3.19}$$

Remark 3.1 S_k is a constant matrix of dimension n^{k-1} by n and determined by P_k. To obtain S_k: first we equally partition $P_k M_k$ as a 1 by n^i block matrix as follows

$$\begin{aligned} P_k M_k \;\; = \;\; & [P_{\underbrace{1\cdots 11}_{i-tuple}} \cdots P_{\underbrace{1\cdots 1n}_{i-tuple}} P_{\underbrace{1\cdots 21}_{i-tuple}} \\ & \cdots P_{\underbrace{1\cdots 2n}_{i-tuple}} \cdots \cdots P_{\underbrace{n\cdots n1}_{i-tuple}} \cdots P_{\underbrace{n\cdots nn}_{i-tuple}}] \end{aligned}$$

and, let

$$P_k^i = \begin{bmatrix} P_{\underbrace{1\cdots 11}_{i-tuple}} & P_{\underbrace{1\cdots 21}_{i-tuple}} & \cdots & P_{\underbrace{n\cdots n1}_{i-tuple}} \\ P_{\underbrace{1\cdots 12}_{i-tuple}} & P_{\underbrace{1\cdots 22}_{i-tuple}} & \cdots & P_{\underbrace{n\cdots n2}_{i-tuple}} \\ \vdots & \vdots & \vdots & \vdots \\ P_{\underbrace{1\cdots 1n}_{i-tuple}} & P_{\underbrace{1\cdots 2n}_{i-tuple}} & \cdots & P_{\underbrace{n\cdots nn}_{i-tuple}} \end{bmatrix},$$

then

$$S_k = \sum_{i=1}^{k} (P_k^i)^T, \quad k = 3, 4, \cdots . \tag{3.20}$$

The above results have shown the coefficients of Taylor series solution for HJI equation can be calculated with one algebraic Ricatti equation and a sequence of linear algebraic equations, which leads to an iterative algorithm to obtain the Taylor series solution of (1.5) as described below. Assuming P, and P_{k+1} for $k = 2, 3, \cdots, N$ are desirable, then the algorithm goes as follows.

Step 1: Solve the Ricatti equation (3.5) for P, obtain $S_2 = P$, and set $k = 2$.

Step 2: Form V_{k+1} and U_{k+1} which depend only on P, P_3, $\cdots$, P_k, and S_2, S_3, $\cdots$, S_k, and obtain P_{k+1} from (3.6) and S_{k+1} from (3.20).

Step 3: If $k = N$ stop. Otherwise $k = k + 1$ and return to step 1.

Overall, the control law (1.6) can be approximated by

$$u = -R_2^{-1} g_2^T(x)(Px + \sum_{k=3}^{\infty} S_k^T N_{k-1} x^{[k-1]}).$$

3.2 Disturbance Attenuation of Flight Control System in Windshears

In this section, we will consider the disturbance attenuation problem associated with flight control system in windshears, which has been introduced in Chapter 1. First, we formulate this control problem as a nonlinear H_∞ control problem.

Letting $V = V_0 + x_1$, $\gamma = x_2$, $\alpha = u$, $z = \begin{bmatrix} \ddot{h} & Mu \end{bmatrix}^T$, $w = [\dot{W}_x \;\; \dot{W}_h]^T$ in equation (1.13) gives

$$\begin{aligned} \dot{x} &= f(x) + g_2(x)u + g_1(x)w \\ z &= h_1(x) + k_{12}(x)u, \end{aligned} \tag{3.21}$$

where

$$
\begin{aligned}
f(x) &= \begin{pmatrix} \frac{a_0+a_1(V_0+x_1)+a_2(V_0+x_1)^2)}{m} - \frac{b_0\rho S(V_0+x_1)^2}{2m} - gx_2 \\ \frac{\delta a_0}{m(V_0+x_1)} + \frac{a_1\delta}{m} + \frac{a_2\delta(V_0+x_1)}{m} + \frac{c_0\rho S(V_0+x_1)}{2m} - \frac{g}{V_0+x_1} \end{pmatrix}, \\
g_1(x) &= \begin{pmatrix} 1 & x_2 \\ \frac{x_2}{V_0+x_1} & \frac{-1}{V_0+x_1} \end{pmatrix}, \\
g_2(x) &= \begin{pmatrix} -\frac{b_1\rho S(V_0+x_1)^2}{2m} \\ \frac{a_0}{m(V_0+x_1)} + \frac{a_1}{m} + \frac{a_2(V_0+x_1)}{m} + \frac{c_1\rho S(V_0+x_1)}{2m} \end{pmatrix}, \\
h_1(x) &= \begin{pmatrix} \frac{\left[a_0+a_1(V_0+x_1)+a_2(V_0+x_1)^2\right](x_2+\delta)}{m} - \frac{b_0\rho S(x_1+V_0)^2 x_2}{2m} + \frac{c_0\rho S(x_1+V_0)^2}{2m} - g \\ 0 \end{pmatrix}, \\
k_{12}(x) &= \begin{pmatrix} 0 \\ M \end{pmatrix},
\end{aligned}
$$

$V_0 > 0$ and δ are chosen so that

$$
f(0) = 0, \;\; h_1(0) = 0, \tag{3.22}
$$

i.e.,

$$
\begin{aligned}
a_0 + a_1V_0 + a_2V_0^2 - \frac{1}{2}b_0\rho SV_0^2 &= 0 \\
(a_0 + a_1V_0 + a_2V_0^2)\delta + \frac{1}{2}c_0\rho Sv_0^2 - mg &= 0.
\end{aligned}
$$

In the following, we use the algorithm developed in section 3.1 to achieve disturbance attenuation for this system.

3.2.1 Design of Control Law

We first find the coefficient matrices of the functions $f(x)$, etc as follows:

$$
\begin{aligned}
A &= A_1 = \begin{bmatrix} \frac{a_1+2a_2V_0-b_0\rho SV_0}{m} & -g \\ -\frac{\delta a_0}{mV_0^2}+\frac{a_2\delta}{m}+\frac{c_0\rho S}{2m}+\frac{g}{V_0^2} & 0 \end{bmatrix}, \\
A_2 &= \begin{bmatrix} (1,1) & (2,1) \\ \frac{a_2}{m}-\frac{b_0\rho S}{2m} & \frac{\delta a_0}{mV_0^3}-\frac{g}{V_0^3} \end{bmatrix}_{2\times 4}, \ A_3 = \begin{bmatrix} (2,1) \\ -\frac{\delta a_0}{mV_0^4}+\frac{g}{V_0^4} \end{bmatrix}_{2\times 8}, \\
B_2 &= B_{21} = \begin{bmatrix} -\frac{b_1\rho SV_0^2}{2m} \\ \frac{a_0}{mV_0}+\frac{a_1}{m}+\frac{a_2V_0}{m}+\frac{c_1\rho SV_0}{2m} \end{bmatrix}, \\
B_{21}^1 &= \begin{bmatrix} -\frac{b_1\rho SV_0}{m} & 0 \\ -\frac{a_0}{mV_0^2}+\frac{a_2}{m}+\frac{c_1\rho S}{2m} & 0 \end{bmatrix}, \ B_{21}^2 = \begin{bmatrix} (1,1) & (2,1) \\ -\frac{b_1\rho S}{2m} & \frac{a_0}{mV_0^3} \end{bmatrix}_{2\times 4}, \\
B_{11} &= \begin{bmatrix} 1 & 0 \end{bmatrix}^T, \ B_{12} = \begin{bmatrix} 0 & -\frac{1}{V_0} \end{bmatrix}^T, \ B_1 = \begin{bmatrix} B_{11} & B_{12} \end{bmatrix}, \\
B_{11}^1 &= \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{V_0} \end{bmatrix}, \ B_{12}^1 = \begin{bmatrix} 0 & 1 \\ \frac{1}{V_0^2} & 0 \end{bmatrix}, \\
B_{11}^2 &= \begin{bmatrix} (2,2) \\ -\frac{1}{V_0^2} \end{bmatrix}_{2\times 4}, \ B_{12}^2 = \begin{bmatrix} (2,1) \\ -\frac{1}{V_0^3} \end{bmatrix}_{2\times 4} \\
C_1 &= \begin{bmatrix} \frac{\delta a_1+2\delta a_2V_0+c_0\rho SV_0}{m} & 0 \\ 0 & 0 \end{bmatrix}, \\
C_2 &= \begin{bmatrix} \frac{\delta a_2+0.5c_0\rho S}{m} & \frac{a_1+2a_2V_0-b_0\rho SV_0}{m} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ C_3 = \begin{bmatrix} (1,2) \\ \frac{a_2-0.5b_0\rho S}{m} \end{bmatrix}_{2\times 8} \\
R_2 &= M^2,
\end{aligned}
$$

where the Taylor series $\frac{1}{V_0+x_1} = \frac{1}{V_0} - \frac{x_1}{V_0^2} + \frac{x_1^2}{V_0^3} - \frac{x_1^3}{V_0^4} + \ldots$ is used.

Let

$$
R = \begin{bmatrix} -\gamma^2 & 0 & 0 \\ 0 & -\gamma^2 & 0 \\ 0 & 0 & R_2 \end{bmatrix},
$$

where γ is the L_2 gain between the output and exogenous signals.

To be specific, we will solve HJI equation up to the 4th order. Choose $\gamma = 2$, we can calculate P from the following algebraic Ricatti equation:

$$
H_{px}^T P + PH_{px} + H_{xx} + PH_{pp}P = 0,
$$

where

$$
B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}, \ H_{px} = A, \ H_{xx} = C_1^T C_1, \ H_{pp} = -BR^{-1}B^T,
$$

$$P = S_2 = \begin{bmatrix} 0.11334 & -6.1447 \\ -6.1447 & 361.47 \end{bmatrix}.$$

Applying (3.11) to (3.19) gives

$$\begin{aligned}
E_3 &= row(S_2A_2), \\
F_2 &= row(C_1^TC_1), \; F_3 = \sum_{l=1}^{2} row(C_l^TC_{3-l}) = row(C_1^TC_2) + row(C_2^TC_1), \\
W_{11}^2 &= row(S_2B_{11}^1), \; W_{12}^2 = row(S_2B_{12}^1), \\
Y_{11}^1 &= B_{11}^TS_2^T, \; Y_{12}^1 = B_{12}^TS_2^T, \\
I_3^1 &= row((W_{11}^2)^TY_{11}^1) + row((W_{12}^2)^TY_{12}^1), \\
Y_{21}^1 &= B_{21}^TS_2^T, \\
W_{21}^2 &= row(S_2B_{21}^1), \\
I_3^2 &= row((W_{21}^2)^TR_2^{-1}Y_{21}^1),
\end{aligned}$$

from which we can calculate V_3 and then P_3 as follows:

$$\begin{aligned}
V_3 &= -(E_3 + \frac{F_3 - 2I_3^2}{2} + \frac{I_3^1}{\gamma^2})N_3, \\
U_3 &= M_3\left(\sum_{i=1}^{3} I_2^{(i-1)} \otimes (H_{px} + H_{pp}P) \otimes I_2^{(3-i)}\right)N_3 \\
&= M_3\Big((H_{px} + H_{pp}P) \otimes I_2^{(2)} + I_2 \otimes (H_{px} + H_{pp}P) \otimes I_2 + I_2^{(2)} \otimes (H_{px} + H_{pp}P)\Big)N_3, \\
P_3 &= V_3U_3^{-1},
\end{aligned}$$

where I_2 is the 2×2 unit matrix. From P_3 we can calculate $S_3 = P_3^1 + P_3^2 + P_3^3$, which will be employed in the computation for solving P_4.

In order to calculate P_4, repeat the algorithm again,

$$\begin{aligned}
E_4 &= \sum_{l=2}^{3} row(S_l A_{4-l+1}) = row(S_2 A_3) + row(S_3 A_2),\\
F_4 &= \sum_{l=1}^{3} row(C_l^T C_{4-l}) = row(C_1^T C_3) + row(C_2^T C_2) + row(C_3^T C_1),\\
Z_4 &= row(S_3 H_{pp} S_3^T),\\
Y_{11}^2 &= B_{11}^T S_3^T,\ \ Y_{12}^2 = B_{12}^T S_3^T,\\
W_{11}^3 &= \sum_{l=2}^{3} row(S_l B_{11}^{4-l}) = row(S_2 B_{11}^2) + row(S_3 B_{11}^1),\\
W_{12}^3 &= \sum_{l=2}^{3} row(S_l B_{12}^{4-l}) = row(S_2 B_{12}^2) + row(S_3 B_{12}^1),\\
I_4^1 &= \sum_{l=2}^{3} row((W_{11}^l)^T Y_{11}^{4-l}) + \sum_{l=2}^{3} row((W_{12}^l)^T Y_{12}^{4-l})\\
&= row((W_{11}^2)^T Y_{11}^2) + row((W_{11}^3)^T Y_{11}^1) + row((W_{12}^2)^T Y_{12}^2) + row((W_{12}^3)^T Y_{12}^1),\\
Y_{21}^2 &= B_{21}^T S_3^T,\\
W_{21}^3 &= \sum_{l=2}^{3} row(S_l B_{21}^{4-l}) = row(S_2 B_{21}^2) + row(S_3 B_{21}^1),\\
I_4^2 &= \sum_{l=2}^{3} row((W_{21}^l)^T R_2^{-1} Y_{21}^{4-l}) = row((W_{21}^2)^T R_2^{-1} Y_{21}^2) + row((W_{21}^3)^T R_2^{-1} Y_{21}^1),\\
G_4^1 &= \sum_{l=2}^{2} row((W_{11}^l)^T W_{11}^{4-l}) + \sum_{l=2}^{2} row((W_{12}^l)^T W_{12}^{4-l}) = row((W_{11}^2)^T W_{11}^2) + row((W_{12}^2)^T W_{12}^2),\\
G_4^2 &= \sum_{l=2}^{2} row((W_{21}^l)^T R_2^{-1} W_{21}^{4-l}) = row((W_{21}^2)^T R_2^{-1} W_{21}^2),
\end{aligned}$$

we can calculate V_4 and then P_4 as follows:

$$\begin{aligned}
V_4 &= -\frac{1}{2}\left(Z_4 + 2E_4 + F_4 + \frac{2I_4^1 + G_4^1}{\gamma^2} - 2I_4^2 - G_4^2\right) N_4,\\
U_4 &= M_4\left(\sum_{i=1}^{4} I_2^{(i-1)} \otimes (H_{px} + H_{pp}P) \otimes I_2^{(4-i)}\right) N_4\\
&= M_4\Big((H_{px} + H_{pp}P) \otimes I_2^{(3)} + I_2 \otimes (H_{px} + H_{pp}P) \otimes I_2^{(2)}\\
&+ I_2^{(2)} \otimes (H_{px} + H_{pp}P) \otimes I_2 + I_2^{(3)} \otimes (H_{px} + H_{pp}P)\Big) N_4,\\
P_4 &= V_4 U_4^{-1}.
\end{aligned}$$

From P_4 we can calculate $S_4 = P_4^1 + P_4^2 + P_4^3 + P_4^4$.

In this case,

$$\begin{aligned} P_3 &= \begin{bmatrix} 0.00013247 & -0.040615 & 2.6694 & -48.816 \end{bmatrix}, \\ P_4 &= \begin{bmatrix} -6.4754 \cdot 10^{-8} & -6.2587 \cdot 10^{-5} & 0.012204 & -0.58759 & 7.9305 \end{bmatrix}. \end{aligned}$$

Since the controller can be expressed as:

$$u = -R_2^{-1} g_2^T(x)(Px + \sum_{k=3}^{\infty} S_k^T N_{k-1} x^{[k-1]}),$$

we can obtain the linear, the second-order, and the third-order terms of the controller as follows:

$$\begin{aligned} u_1 &= -R_2^{-1} B_2^T P x, \\ u_2 &= -R_2^{-1} B_2^T S_3^T N_2 x^{[2]} - R_2^{-1} \left(g_2^{[1]}(x) \right)^T P x \\ &= -R_2^{-1} B_2^T S_3^T N_2 x^{[2]} - R_2^{-1} \left(x^{(1)} \right)^T (B_{21}^1)^T P x^{(1)} \\ &= -R_2^{-1} B_2^T S_3^T N_2 x^{[2]} - R_2^{-1} row\left((B_{21}^1)^T P \right) x^{(2)} \\ &= -R_2^{-1} \left(B_2^T S_3^T + row\left((B_{21}^1)^T P \right) \right) N_2 x^{[2]} \\ u_3 &= -R_2^{-1} B_2^T S_4^T N_3 x^{[3]} - R_2^{-1} \left(g_2^{[1]}(x) \right)^T S_3^T N_2 x^{[2]} - R_2^{-1} \left(g_2^{[2]}(x) \right)^T P x \\ &= -R_2^{-1} B_2^T S_4^T N_3 x^{[3]} - R_2^{-1} \left(x^{(1)} \right)^T (B_{21}^1)^T S_3^T x^{(2)} - R_2^{-1} \left(x^{(2)} \right)^T (B_{21}^2)^T P x^{(1)} \\ &= -R_2^{-1} B_2^T S_4^T N_3 x^{[3]} - R_2^{-1} row\left((B_{21}^1)^T S_3^T \right) x^{(3)} - R_2^{-1} row\left((B_{21}^2)^T P \right) x^{(3)} \\ &= -R_2^{-1} \left(B_2^T S_4^T + row\left((B_{21}^1)^T S_3^T \right) + row\left((B_{21}^2)^T P \right) \right) N_3 x^{[3]} \end{aligned}$$

where (3.3) is used.

Then the linear approximation controller is $u = u_1$, the second order approximation controller is $u = u_1 + u_2$, and the third order approximation controller is $u = u_1 + u_2 + u_3$, where in this case

$$\begin{aligned} u_1 &= [\ 0.080228 \quad -10.059\ ] x, \\ u_2 &= [\ 9.5714 \cdot 10^{-5} \quad -0.045953 \quad 2.2376\ ] x^{[2]}, \\ u_3 &= [\ 4.3227 \cdot 10^{-8} \quad 8.6562 \cdot 10^{-6} \quad 0.012087 \quad -0.38878\ ] x^{[3]}. \end{aligned}$$

The controller design of this section repeats the iterative algorithm presented in the end of last section. Besides of P, the rest coefficients are all achieved via some simple linear matrix operations. That is to say, the approximation algorithm has bypassed the difficult computation of PDE and has converted the complex nonlinear control problem to some simple and systematic linear algebraic calculations. The advantage of the algorithm also lies in the simplicity of programming. The programs of this part have been attached in Appendix.

3.2.2 Computer Simulation

It has been shown that the algorithm is systematic and simple to follow and program. In this section, we evaluate the controllers achieved by this method. We will compare the stabilization and disturbance attenuation properties of linear controller, the second order nonlinear controller and the third order nonlinear controller for aircraft flight system in windshears. The horizontal and vertical speed of the windshear is given by [26]

$$\begin{aligned} W_x &= -W_{x0} sin(\frac{2\pi t}{T_0}) \\ W_h &= -W_{h0}\left(1 - cos(\frac{2\pi t}{T_0})\right) \end{aligned}$$

and T_0 is the design parameter. The parameters and constants we use for the simulation are similar with [26] and [27].

With initial condition $x(0) = [\ -10 \quad 0.1\ ]$, Figure 3.1 shows the system behaviors under different controllers without disturbance. All these three controllers are able to stabilize the closed-loop system.

Choosing

$$T_0 = 60(sec),\ \ W_{x0} = 50(ft/sec),\ \ W_{h0} = 30(ft/sec),$$

gives

$$w_1 = \begin{bmatrix} -\frac{5\pi}{3} cos(\frac{\pi t}{30}) \\ -\pi sin(\frac{\pi t}{30}) \end{bmatrix}.$$

And choosing $T_0 = 60, W_{x0} = 40, 0, 50$, and $W_{h0} = 25, 30, 0$ respectively, gives

$$w_2 = \begin{bmatrix} -\frac{4\pi}{3} cos(\frac{\pi t}{30}) \\ -\frac{5\pi}{6} sin(\frac{\pi t}{30}) \end{bmatrix},\ w_3 = \begin{bmatrix} 0 \\ -\pi sin(\frac{\pi t}{30}) \end{bmatrix},\ w_4 = \begin{bmatrix} -\frac{5\pi}{3} cos(\frac{\pi t}{30}) \\ 0 \end{bmatrix}.$$

With initial condition $x(0) = [\ 0 \quad 0\ ]$, Figure 3.2 to 3.5 shows the system behaviors with disturbance $w = w_1, w_2, w_3$ or w_4 respectively, and Figure 3.6 repeats the scenario of Figure 3.2 but with disturbance $w = 10w_1$, under the linear controller, the second-order controller and the third-order controller. Figure 3.7 shows the system behaviors with disturbance $w = 15w_1$, for only the linear and the second-order controllers. From the figures, we can tell, with the small sinusoidal disturbance in Figure 3.2 to 3.5, three controllers have almost the same performances, but for the large sinusoidal disturbance, the second order is the best, and the third order controller is the worst. Especially, in Figure 3.6, the third order nonlinear controller could not keep the

Frequency	3nd order controller	2nd order controller	Linear controller
$\omega = \frac{\pi}{3}$	divergence	30.989	32.318
$\omega = \frac{\pi}{2}$	24.697	22.907	24.091
$\omega = \frac{2\pi}{3}$	17.971	17.522	18.582

Table 3.1: Maximal steady state amplitudes of $\ddot{h}$ with $A_m = 13[-\frac{5}{3}\pi \quad -\pi]^T$.

Amplitude	3nd order controller	2nd order controller	Linear controller
$A_m = 9[-\frac{5}{3}\pi \quad -\pi]^T$	20.435	19.821	20.644
$A_m = 13[-\frac{5}{3}\pi \quad -\pi]^T$	divergence	30.989	32.318
$A_m = 17[-\frac{5}{3}\pi \quad -\pi]^T$	divergence	42.779	divergence

Table 3.2: Maximal steady state amplitudes of $\ddot{h}$ with $\omega = \frac{\pi}{3}$.

system bounded, but the rest two controllers can achieve a bounded motion. And in Figure 3.7, only the second order controller can achieve a bounded motion.

To demonstrate more scenarios, Table 3.1 lists the steady state amplitudes of the time response of the vertical acceleration $\ddot{h}$ for several different disturbance frequencies with $A_m = 13[-\frac{5}{3}\pi \quad -\pi]^T$ and Table 3.2 lists the steady state amplitudes of the time response of the vertical acceleration $\ddot{h}$ for several different disturbance amplitudes with $\omega = \frac{\pi}{3}$. It can be seen that when the amplitude of the disturbance is relatively small, the performances of the linear, second-order and third-order controllers are similar. But as the amplitude of the disturbance increases, the performance of the second-order controller is better than the linear and the third-order controllers.

3.3 Conclusions

In this section, we have applied an approximation method to design continuous-time nonlinear H_∞ controllers for the flight control system in windshears. The outcome of the investigation shows, on one hand, that the complex algorithm developed in [24] can be successfully implemented by programming softwares such as Matlab, and on the other hand, that a higher order approximation of nonlinear H_∞control law may perform better than a linear H_∞ controller, a conclusion that may not be taken for granted.

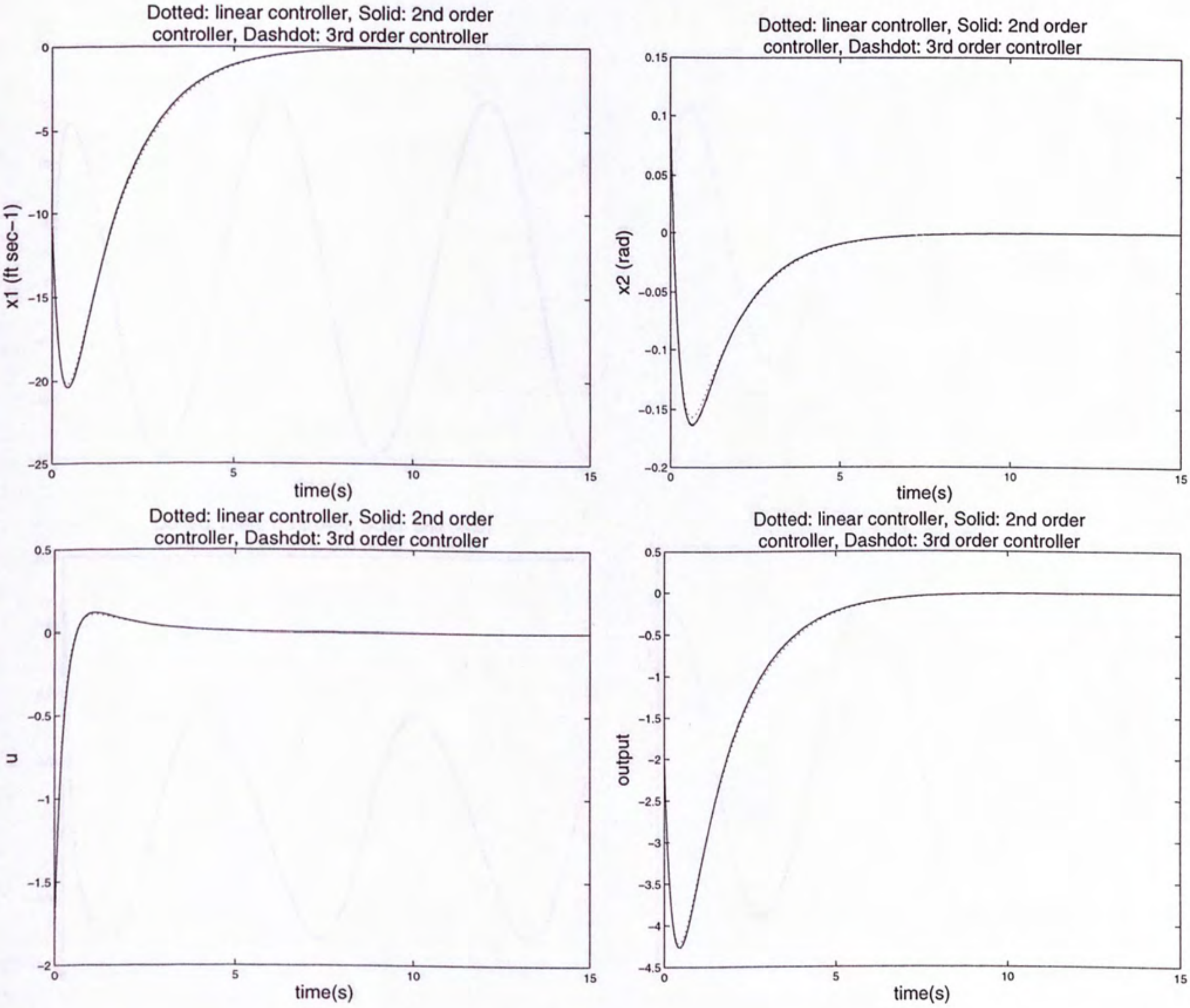

Figure 3.1: The states, input, and output behavior comparisons between the linear controller, the second order controller and the third order controller, with no-zero initial states and without disturbance.

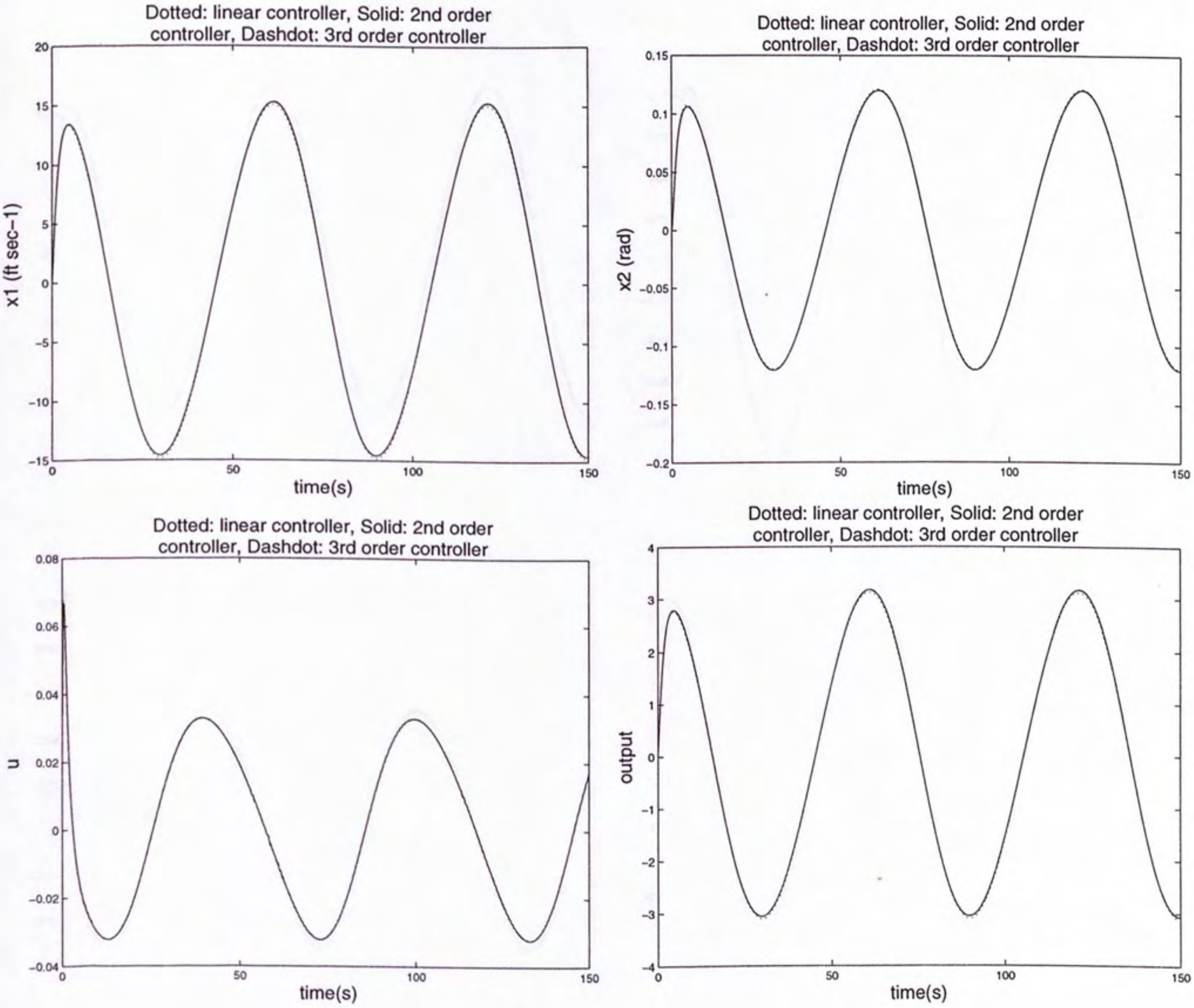

Figure 3.2: The states, input, and output behavior comparisons between the linear controller, the second order controller and the third order controller, with zero initial states and disturbance $w = w_1$.

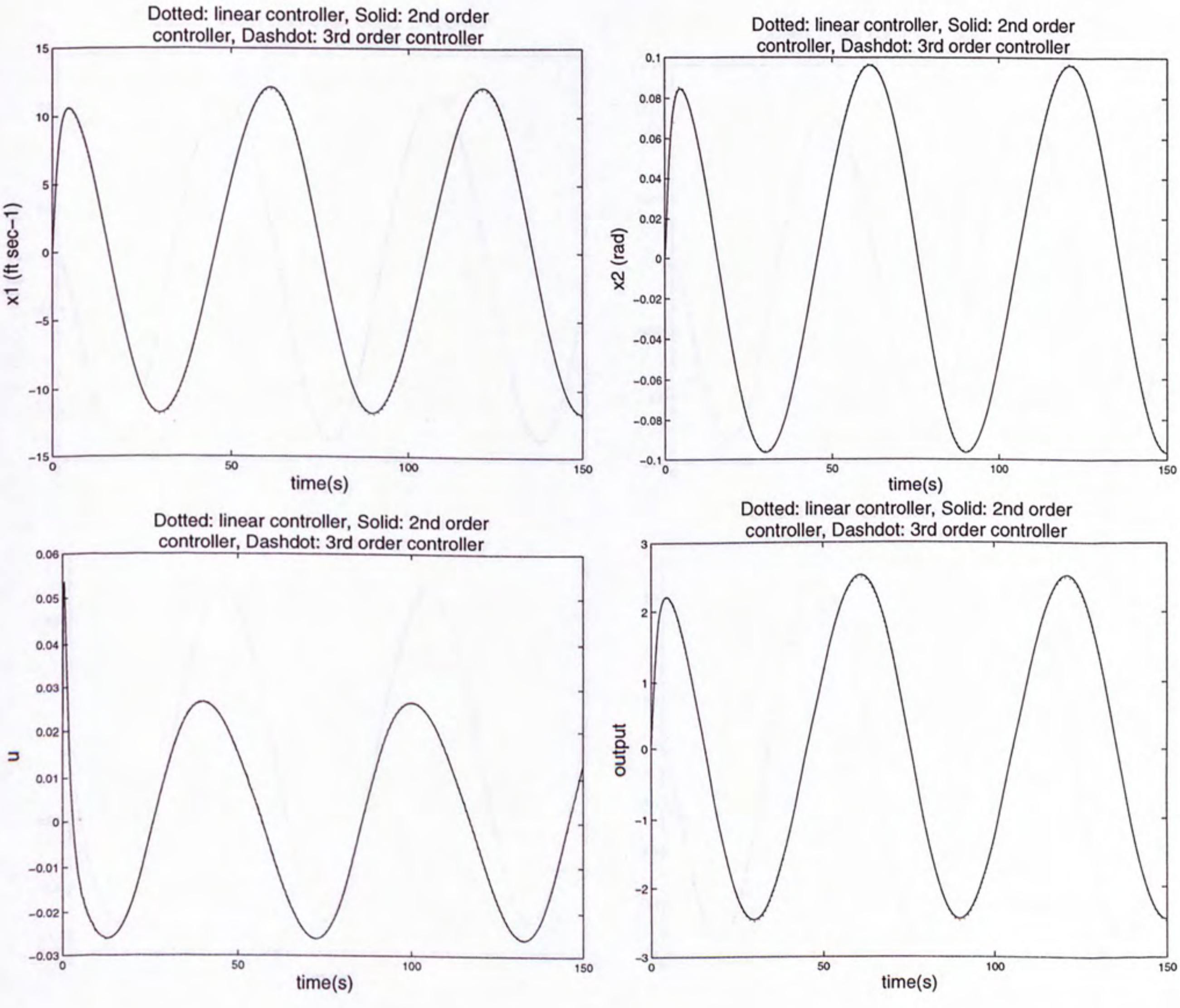

Figure 3.3: The states, input, and output behavior comparisons between the linear controller, the second order controller and the third order controller, with zero initial states and disturbance $w = w_2$.

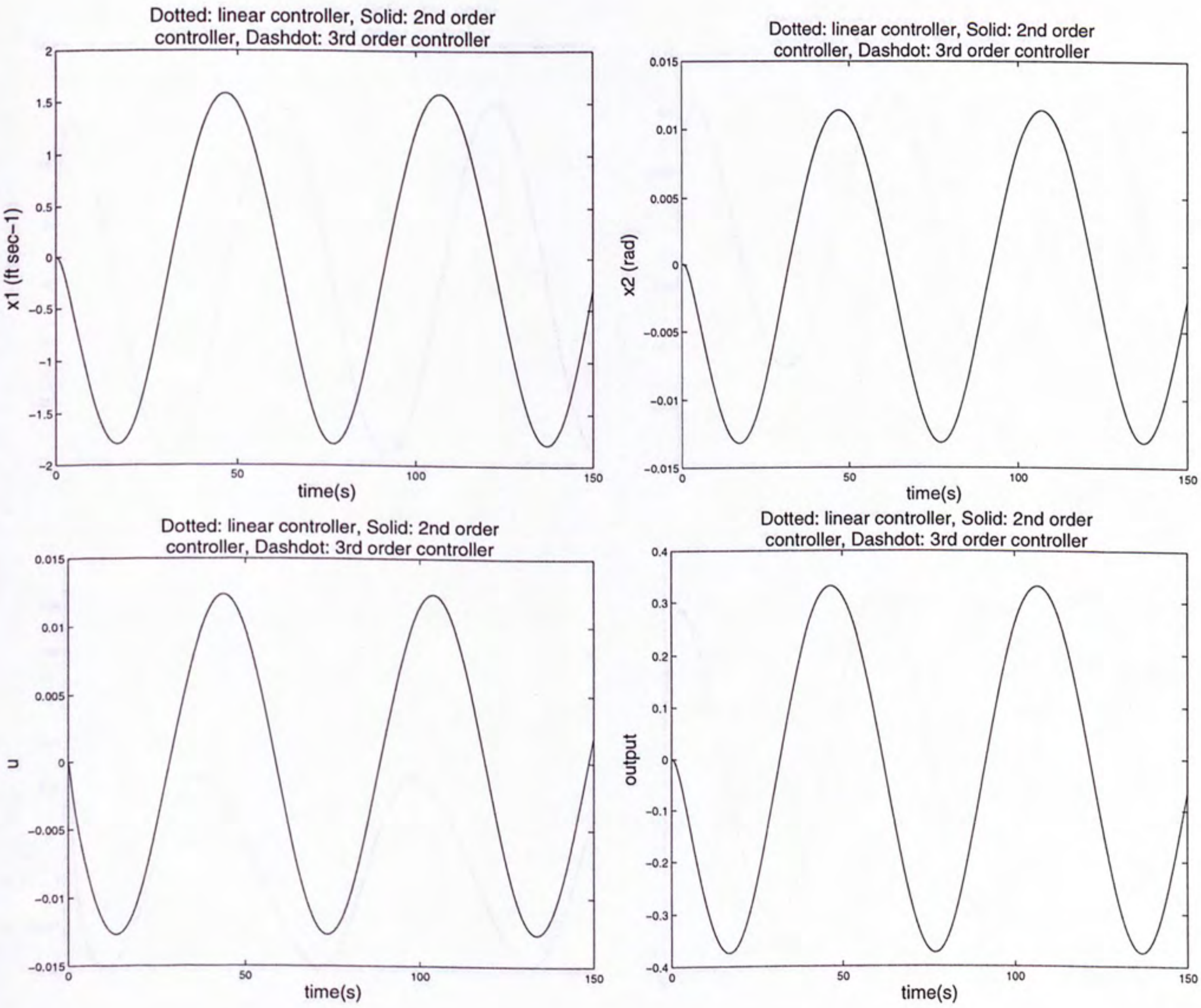

Figure 3.4: The states, input, and output behavior comparisons between the linear controller, the second order controller and the third order controller, with zero initial states and disturbance $w = w_3$.

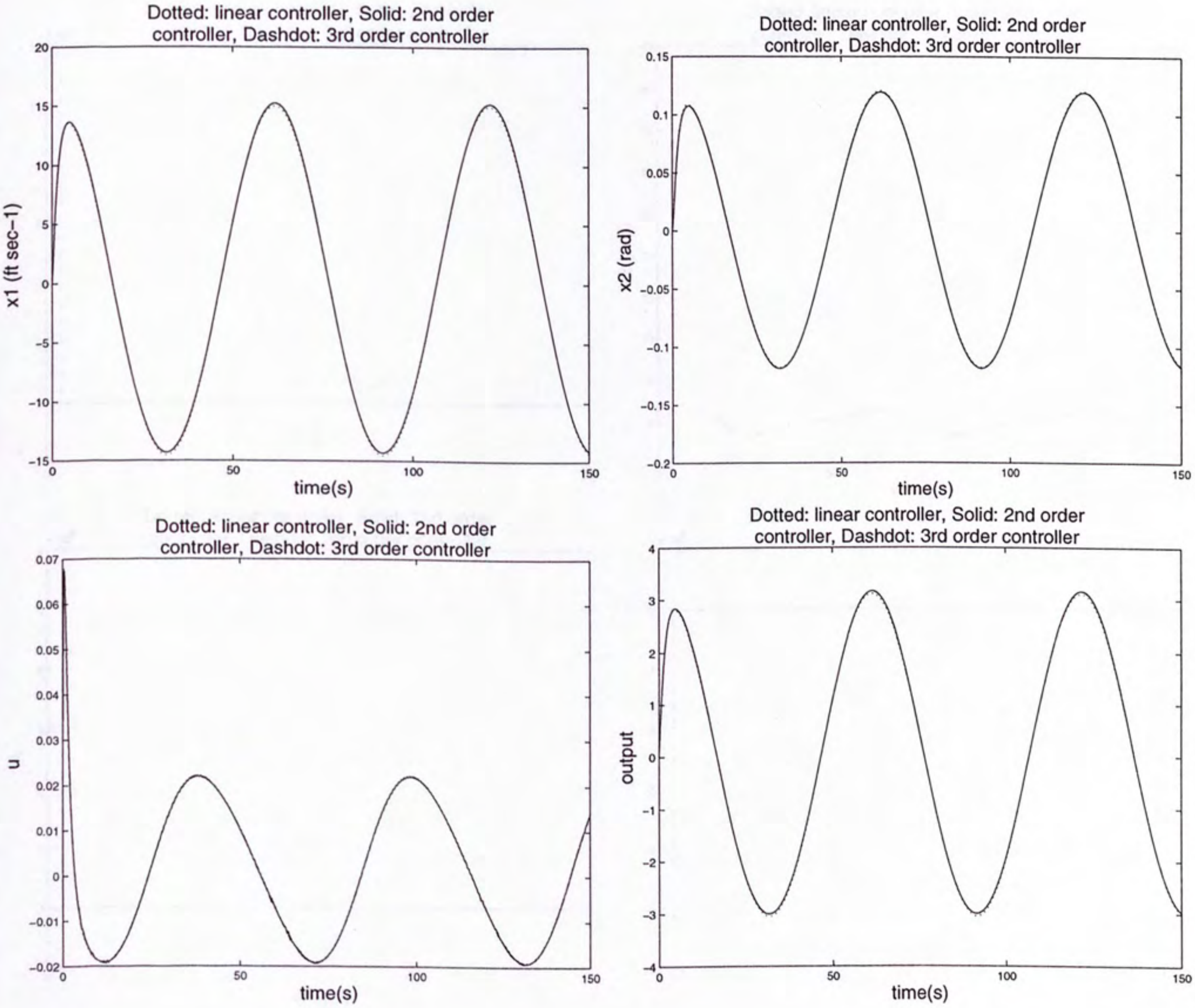

Figure 3.5: The states, input, and output behavior comparisons between the linear controller, the second order controller and the third order controller, with zero initial states and disturbance $w = w_4$.

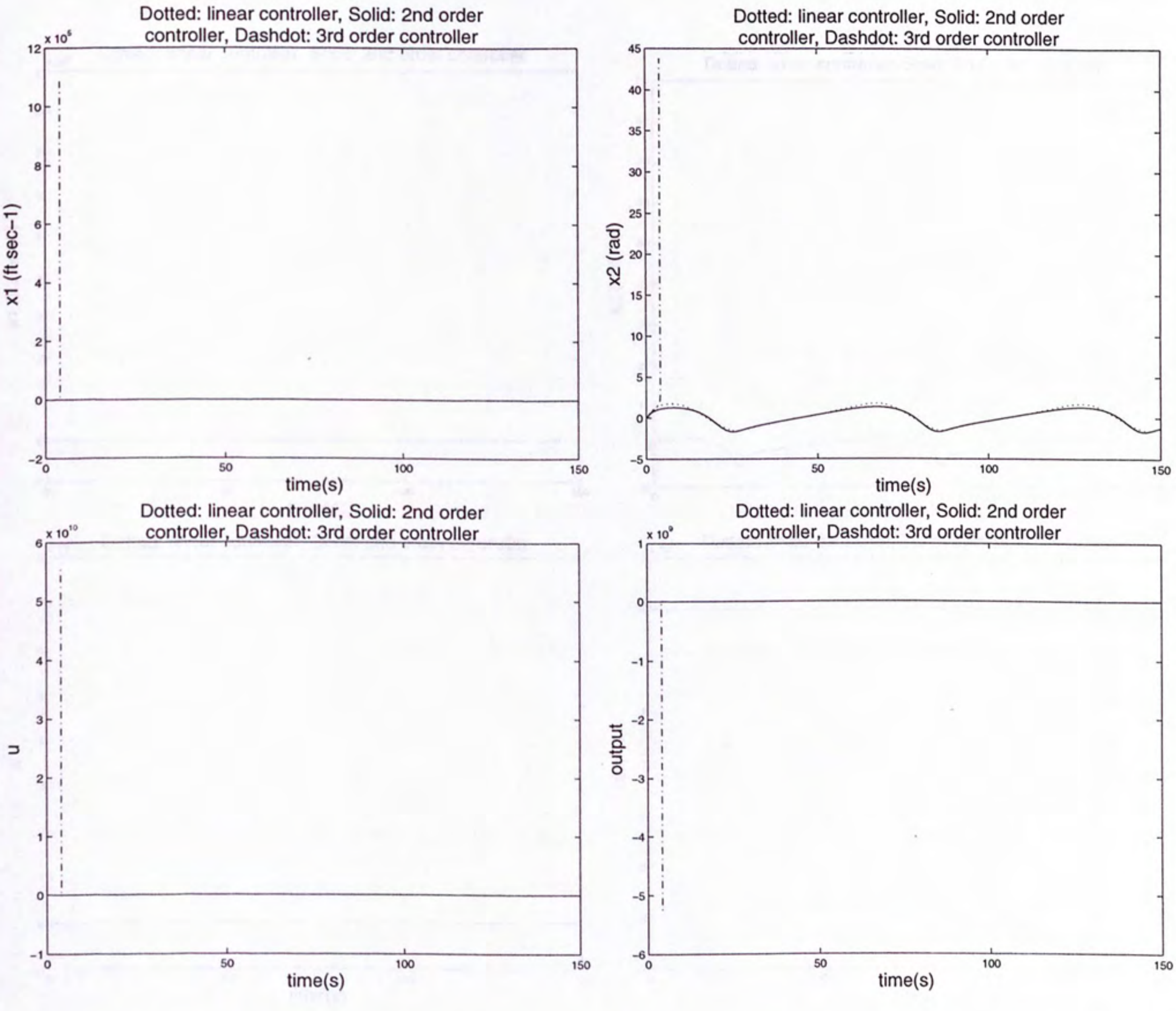

Figure 3.6: The states, input, and output behavior comparisons between the linear controller, the second order controller and the third order controller, with zero initial states and disturbance $w = 10w_1$.

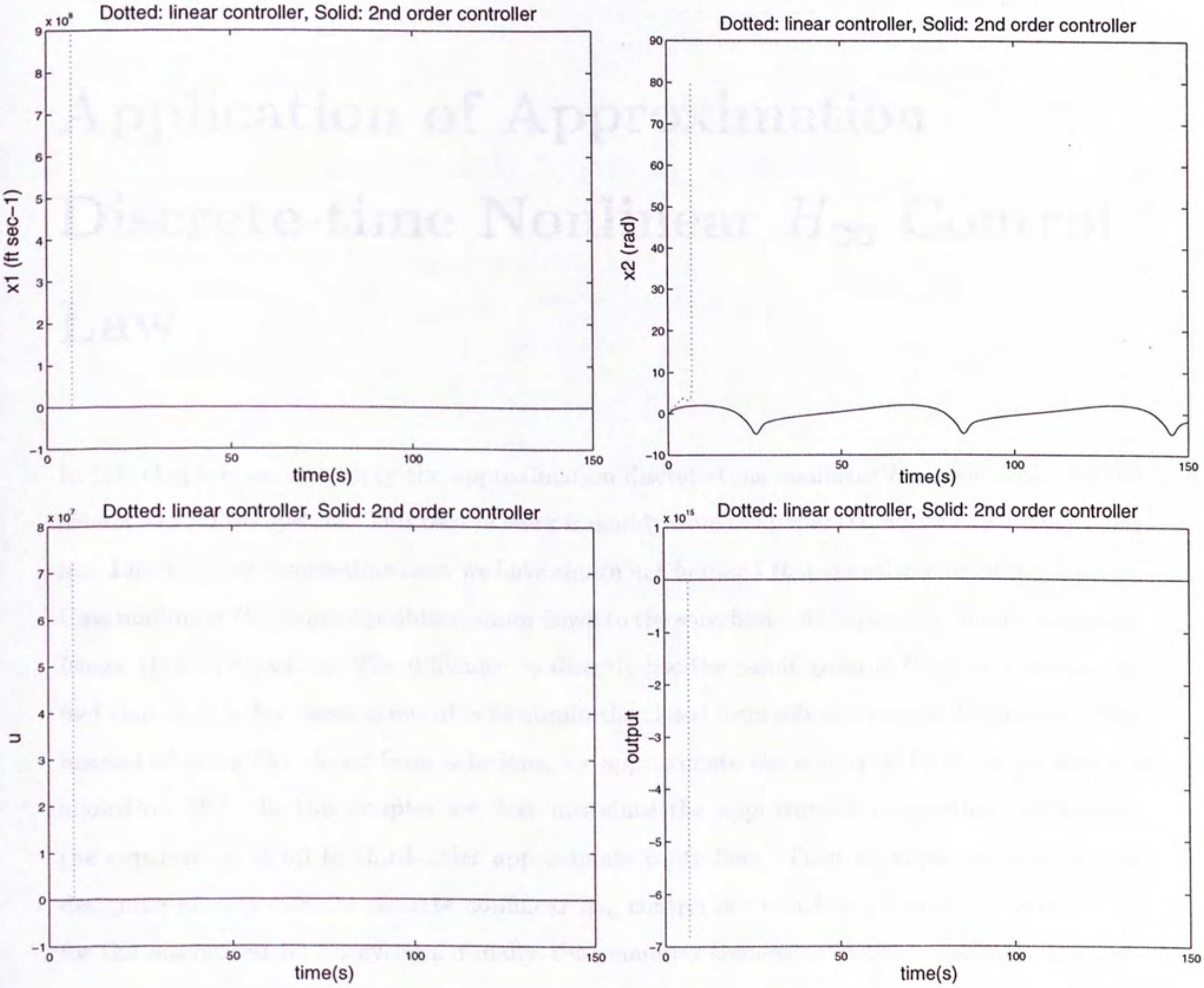

Figure 3.7: The states, input, and output behavior comparisons between the linear controller and the second order controller, with zero initial states and disturbance $w = 15w_1$.

Chapter 4

Application of Approximation Discrete-time Nonlinear H_∞ Control Law

In this chapter, we will apply the approximation discrete-time nonlinear H_∞ control law for the discretized RTAC system. This part of work is mainly from the paper [31] written by J.Huang and me. Like the continuous-time case, we have shown in Chapter 1 that the solvability of the discrete-time nonlinear H_∞ control problem comes down to the solvability of the discrete Hamilton-Jacobi-Isaacs (DHJI) equation. The difficulty to directly use the result given in Chapter 1 lies on the fact that only in few cases is one able to obtain the closed form solution for DHJI equation. Thus instead of using the closed form solutions, we approximate the controller (1.12) in an iterative algorithm [23]. In this chapter we first introduce the approximation algorithm and present the expressions of up to third-order approximate controllers. Then we apply the approach to designing an approximate discrete nonlinear H_∞ control law to achieve disturbance attenuation for the discretized RTAC system. Finally, the computer simulation and the conclusion is given.

4.1 Preliminaries of Approximation Discrete-time Nonlinear H_∞ Control Law

Using the same notations as those in Chapter 3 gives the following unique expressions for the Taylor series expansions $V(x), u^*(x), w^*(x)$:

$$V(x) = \frac{1}{2}x^T P x + \sum_{k=3}^{\infty} P_k x^{[k]} \tag{4.1}$$

$$u^*(x) = \sum_{k=1}^{\infty} U_k x^{[k]}, \quad w^*(x) = \sum_{k=1}^{\infty} W_k x^{[k]} \tag{4.2}$$

where P is $n \times n$ symmetric matrix and P_k, U_k, W_k are row vectors.

The following result is from [23].

Theorem 4.1 Consider the system (1.7) with $m = r = p = 1$. Let the Taylor series representations of $A(x), B(x), E(x), C(x), D(x)$, and $F(x)$ be given by

$$\begin{aligned} A(x) &= Ax + \sum_{k=2}^{\infty} A_k x^{[k]}, \quad B(x) = B + \sum_{k=1}^{\infty} B_k x^{[k]}, \\ E(x) &= E + \sum_{k=1}^{\infty} E_k x^{[k]}, \quad C(x) = Cx + \sum_{k=2}^{\infty} C_k x^{[k]}, \\ D(x) &= D + \sum_{k=1}^{\infty} D_k x^{[k]}, \quad F(x) = F + \sum_{k=1}^{\infty} F_k x^{[k]} \end{aligned} \tag{4.3}$$

Then the coefficient vectors of the Taylor series expansions of $u^*(x)$, $w^*(x)$ and $V(x)$ are given by

$$\begin{bmatrix} U_1 \\ W_1 \end{bmatrix} = -R^{-1} \begin{bmatrix} B^T PA + D^T C \\ E^T PA + F^T C \end{bmatrix} \tag{4.4}$$

where

$$R = \begin{bmatrix} B^T PB + D^T D & B^T PE + D^T F \\ E^T PB + F^T D & E^T PE + F^T F - \gamma^2 \end{bmatrix} \tag{4.5}$$

and P is governed by the following discrete-time algebraic Ricatti equation

$$\begin{aligned} P &= A^T PA + C^T C \\ &\quad - \begin{bmatrix} B^T PA + D^T C \\ E^T PA + F^T C \end{bmatrix}^T R^{-1} \begin{bmatrix} B^T PA + D^T C \\ E^T PA + F^T C \end{bmatrix} \end{aligned} \tag{4.6}$$

and, for $k \geq 2$,

$$\begin{bmatrix} U_k \\ W_k \end{bmatrix} = \begin{bmatrix} P_{k+1}\xi_k^u + \eta_k^u \\ P_{k+1}\xi_k^w + \eta_k^w \end{bmatrix} \tag{4.7}$$

$$P_{k+1} L_{k+1} = H_{k+1} \tag{4.8}$$

where, for $k \geq 2$, ξ_k^u, ξ_k^w, η_k^u, and η_k^w depend only on P, P_2, $\cdots$, P_k, and (U_1, W_1), $\cdots$, (U_{k-1}, W_{k-1}), and L_{k+1} and H_{k+1} depend only on $P, P_3, \cdots, P_k$, and $(U_1, W_1), \cdots, (U_{k-1}, W_{k-1})$. Explicit expressions for ξ_k^u, ξ_k^w, η_k^u, and η_k^w as well as L_{k+1} and H_{k+1} can be found in [23] as follows:

$$\begin{bmatrix} \xi_k^u \\ \xi_k^w \end{bmatrix} = \begin{bmatrix} M_{k+1}(r_{11}B_{k+1}^k + r_{12}E_{k+1}^k)N_k \\ M_{k+1}(r_{21}B_{k+1}^k + r_{22}E_{k+1}^k)N_k \end{bmatrix} \tag{4.9}$$

$$\begin{bmatrix} \eta_k^u \\ \eta_k^w \end{bmatrix} = -R^{-1} \begin{bmatrix} G_k^u \\ G_k^w \end{bmatrix} N_k \tag{4.10}$$

$$\begin{aligned} L_k &= M_k \phi_1^{(k)} N_k - I_n^{[k]} + ((\phi_1^T P B + \psi_1^T D)^T \otimes \xi_{k-1}^u M_{k-1} \\ &\quad + (\phi_1^T P E + \psi_1^T F - \gamma^2 W_1^T)^T \otimes \xi_{k-1}^w M_{k-1}) N_k \end{aligned} \tag{4.11}$$

$$\begin{aligned} H_k &= -\frac{1}{2} \sum_{\substack{i+j=k \\ i,j\geq 2}} row(\phi_i^T P \phi_j + \psi_i^T \psi_j - \gamma^2 M_i^T W_i^T W_j M_j) N_k - \sum_{j=3}^{k-1} P_j M_j \Gamma_{kj} N_k \\ &- row\left(\begin{bmatrix} P\phi_1 \\ \psi_1 \\ -\gamma^2 W_1 \end{bmatrix}^T \begin{bmatrix} (B\eta_{k-1}^u + E\eta_{k-1}^w)M_{k-1} + A_{k-1} + B_{k-1}^u + E_{k-1}^w \\ (D\eta_{k-1}^u + F\eta_{k-1}^w)M_{k-1} + C_{k-1} + D_{k-1}^u + F_{k-1}^w \\ \eta_{k-1}^w M_{k-1} \end{bmatrix} \right) N_k \end{aligned} \tag{4.12}$$

where,

$$\begin{bmatrix} G_k^u \\ G_k^w \end{bmatrix} = \begin{bmatrix} B^T P(A_k + B_k^u + E_k^w) + D^T(C_k + D_k^u + F_k^w) \\ E^T P(A_k + B_k^u + E_k^w) + F^T(C_k + D_k^u + F_k^w) \end{bmatrix} + \begin{bmatrix} \sum_{\substack{i+j=k \\ i,j\geq 1}} row(B_i^T P\phi_j + D_i^T \psi_j) + \sum_{j=3}^k P_j M_j B_j^k \\ \sum_{\substack{i+j=k \\ i,j\geq 1}} row(E_i^T P\phi_j + F_i^T \psi_j) + \sum_{j=3}^k P_j M_j E_j^k \end{bmatrix} \tag{4.13}$$

$$B_k^j = \sum_{\substack{t+s=j \\ s\geq 0, t\geq k-1}} B_k^{ts}, \quad j \geq k-1 \tag{4.14}$$

$$B_k^{ts} = \sum_{i=1}^{k} \Big(\sum_{\substack{l+m=t \\ l\geq i-1, m\geq k-i}} \Gamma_{l(i-1)} \otimes B_s \otimes \Gamma_{m(k-i)} \Big), \quad s \geq 0 \tag{4.15}$$

$$E_k^j = \sum_{\substack{t+s=j \\ s\geq 0, t\geq k-1}} E_k^{ts}, \quad j \geq k-1 \tag{4.16}$$

$$E_k^{ts} = \sum_{i=1}^{k} \Big(\sum_{\substack{l+m=t \\ l\geq i-1, m\geq k-i}} \Gamma_{l(i-1)} \otimes E_s \otimes \Gamma_{m(k-i)} \Big), \quad s \geq 0 \tag{4.17}$$

$$\Gamma_{kj} = \begin{cases} 0_{1\times n^k}, & j = 0, k > 0 \\ 1, & k = j = 0 \\ \sum_{\substack{i_1+i_2+\cdots+i_j=k \\ i_1,i_2,\cdots,i_j\geq 1}} \phi_{i_1} \otimes \phi_{i_2} \otimes \cdots \otimes \phi_{i_j}, & k \geq j \geq 1 \end{cases} \tag{4.18}$$

and for $k \geq 1$,

$$\phi_k = BU_kM_k + EW_kM_k + (A_k + B_k^u + E_k^w)$$
$$\psi_k = DU_kM_k + FW_kM_k + (C_k + D_k^u + F_k^w)$$

$$B_k^u = \begin{cases} 0, & k = 1 \\ \sum_{\substack{i+j=k \\ i,j\geq 1}} B_i(I_n^{(i)} \otimes U_jM_j), & k > 1 \end{cases}$$

$$E_k^w = \begin{cases} 0, & k = 1 \\ \sum_{\substack{i+j=k \\ i,j\geq 1}} E_i(I_n^{(i)} \otimes W_jM_j), & k > 1 \end{cases}$$

$$D_k^u = \begin{cases} 0, & k = 1 \\ \sum_{\substack{i+j=k \\ i,j\geq 1}} D_i(I_n^{(i)} \otimes U_jM_j), & k > 1 \end{cases}$$

$$F_k^w = \begin{cases} 0, & k = 1 \\ \sum_{\substack{i+j=k \\ i,j\geq 1}} F_i(I_n^{(i)} \otimes W_jM_j), & k > 1 \end{cases} \tag{4.19}$$

In (4.15) and (4.17), $B_0 \stackrel{def}{=} B$ and $E_0 \stackrel{def}{=} E$.

The above theorem leads to an iterative algorithm to obtain the Taylor series solution of (1.10) to (1.9) as described below. Assuming P, and (U_k, W_k, P_{k+1}) for $k = 2, 3, \cdots, N$ are desirable, then the algorithm goes as follows.

Step 1: Solve the discrete Ricatti equation (4.6) for P, obtain U_1 and W_1 from equation (4.4), and set $k = 2$.

Step 2: Form L_{k+1} and H_{k+1} which depend only on P, P_3, $\cdots$, P_k, and (U_1, W_1), $\cdots$, (U_{k-1}, W_{k-1}), and obtain P_{k+1} from (4.8).

Step 3: Form ξ_k^u, ξ_k^w, η_k^u, and η_k^w which depend only on P, P_3, $\cdots$, P_k, and (U_1, W_1), $\cdots$, (U_{k-1}, W_{k-1}), and obtain U_k and W_k from (4.7).

Step 4: If $k = N$ stop. Otherwise $k = k + 1$ and return to step 1.

It is seen that, in a way similar to the continuous case, the problem of approximating the discrete nonlinear H_∞ control laws is reduced to solving one discrete algebraic Ricatti equation and a sequence of linear algebraic equations.

4.2 Explicit Expression of $u^{[k]}$

In Section 4.1, we have obtained P_k, U_k and W_k. But the approximation controller can not be directly expressed with these parameters. We will find the explicit expression of $u^{[k]}$ in this section. Here, we just give the expression of up to third-order controller, the higher-order controller can be generated with the same method.

To obtain the third order controller from (1.12), we need to calculate $H_{uu}(x, u^*, w^*)$ and $H_{wu}(x, u^*, w^*)$ up to the second order using the following formulas

$$H_{uu}(x, u^*, w^*) = B^T(x) \left(\frac{\partial^2 V}{\partial \alpha^2}\right)\bigg|_{\alpha = A(x)+B(x)u^*+E(x)w^*} B(x) + D^T(x)D(x),$$

$$H_{wu}(x, u^*, w^*) = E^T(x) \left(\frac{\partial^2 V}{\partial \alpha^2}\right)\bigg|_{\alpha = A(x)+B(x)u^*+E(x)w^*} B(x) + F^T(x)D(x),$$

We first look for the second order term of α in $\frac{\partial^2 V}{\partial \alpha^2}|_{\alpha=A(x)+B(x)u^*+E(x)w^*}$,

$$\begin{aligned}
\frac{\partial^2 V}{\partial \alpha^2} &= P + \frac{\partial^2(P_3 M_3 \alpha^{(3)}) + P_4 M_4 \alpha^{(4)})}{\partial \alpha^2} \\
&= P + \frac{\partial \left[P_3 M_3 \Sigma_{i=1}^{3}(\alpha^{(i-1)} \otimes I_4 \otimes \alpha^{(3-i)})\right]^T}{\partial \alpha} + \frac{\partial \left[P_4 M_4 \Sigma_{i=1}^{4}(\alpha^{(i-1)} \otimes I_4 \otimes \alpha^{(4-i)})\right]^T}{\partial \alpha} \\
&= P + \frac{\partial \left[(\alpha^T)^{(2)} S_3\right]^T}{\partial \alpha} + \frac{\partial \left[(\alpha^T)^{(3)} S_4\right]^T}{\partial \alpha} \\
&= P + \frac{\partial \left[S_3^T \alpha^{(2)}\right]}{\partial \alpha} + \frac{\partial \left[S_4^T \alpha^{(3)}\right]}{\partial \alpha} \\
&= P + \left[S_3^T \Sigma_{i=1}^{2}(\alpha^{(i-1)} \otimes I_4 \otimes \alpha^{(2-i)})\right]^T + \left[S_4^T \Sigma_{i=1}^{3}(\alpha^{(i-1)} \otimes I_4 \otimes \alpha^{(3-i)})\right]^T \\
&= P + (\alpha^{(0)} \otimes I_4 \otimes \alpha^{(1)} + \alpha^{(1)} \otimes I_4 \otimes \alpha^{(0)})^T S_3 \\
&\quad + (\alpha^{(0)} \otimes I_4 \otimes \alpha^{(2)} + \alpha^{(1)} \otimes I_4 \otimes \alpha^{(1)} + \alpha^{(2)} \otimes I_4 \otimes \alpha^{(0)})^T S_4 \\
&= P + \bar{S}_3(\alpha) S_3 + \bar{S}_4(\alpha) S_4
\end{aligned}$$

where the identity (3.2) is applied, S_3 and S_4 are defined in (3.20) and uniquely determined by P_3 and P_4, and

$$\begin{aligned}
\bar{S}_3(\alpha) &= (\alpha^{(0)} \otimes I_4 \otimes \alpha^{(1)} + \alpha^{(1)} \otimes I_4 \otimes \alpha^{(0)})^T \\
\bar{S}_4(\alpha) &= (\alpha^{(0)} \otimes I_4 \otimes \alpha^{(2)} + \alpha^{(1)} \otimes I_4 \otimes \alpha^{(1)} + \alpha^{(2)} \otimes I_4 \otimes \alpha^{(0)})^T
\end{aligned} \tag{4.20}$$

Also, we have

$$\begin{aligned}
\alpha &= \Sigma_{k=1}^{\infty} \phi_k x^{(k)} = \phi_1 x + \phi_2 x^{(2)} + O^{[3+]}(x), \\
\alpha^{(2)} &= (\Sigma_{k=1}^{\infty} \phi_k x^{(k)}) \otimes (\Sigma_{k=1}^{\infty} \phi_k x^{(k)}) \\
&= \phi_1^{(2)} x^{(2)} + O^{[3+]}(x),
\end{aligned}$$

where $O^{[3+]}(x)$ denotes any smooth function which vanishes at the origin together with its first and second order derivatives.

Using identities (3.3) and (3.3) gives

$$\begin{aligned}
B^T \bar{S}_4(\alpha) S_4 B &= B^T (\alpha^{(0)} \otimes I_4 \otimes \alpha^{(2)} + \alpha^{(1)} \otimes I_4 \otimes \alpha^{(1)} + \alpha^{(2)} \otimes I_4 \otimes \alpha^{(0)})^T S_4 B \\
&= \left[(\alpha^{(0)} \otimes I_4 \otimes \alpha^{(2)}) B \alpha^{(0)} + (\alpha^{(1)} \otimes I_4 \otimes \alpha^{(1)}) B \alpha^{(0)} + (\alpha^{(2)} \otimes I_4 \otimes \alpha^{(0)}) B \alpha^{(0)}\right]^T S_4 B \\
&= \left[(I_4^{(0)} \otimes B \otimes I_4^{(2)}) \alpha^{(2)} + (I_4^{(1)} \otimes B \otimes I_4^{(1)}) \alpha^{(2)} + (I_4^{(2)} \otimes B \otimes I_4^{(0)}) \alpha^{(2)}\right]^T S_4 B \\
&= B^T S_4^T \left[(I_4^{(0)} \otimes B \otimes I_4^{(2)}) + (I_4^{(1)} \otimes B \otimes I_4^{(1)}) + (I_4^{(2)} \otimes B \otimes I_4^{(0)})\right] \alpha^{(2)} \\
&= B^T S_4^T \bar{B}_4 \alpha^{(2)},
\end{aligned}$$

$$
\begin{aligned}
x^T B_1^T \bar{S}_3(\alpha) S_3 B &= B^T S_3^T \left[I_4 \otimes (\phi_1 x) + (\phi_1 x) \otimes I_4 \right] B_1 x + O^{[3+]} \\
&= B^T S_3^T \left[(I_4 \otimes \phi_1)(I_4 \otimes x) + (\phi_1 \otimes I_4)(x \otimes I_4) \right] B_1 x + O^{[3+]} \\
&= B^T S_3^T \Big[(I_4 \otimes \phi_1)(x^{(0)} \otimes I_4 \otimes x) B_1 x + (\phi_1 \otimes I_4)(x \otimes I_4 \otimes x^{(0)}) B_1 x \Big] + O^{[3+]} \\
&= B^T S_3^T \Big[(I_4 \otimes \phi_1)(I_4^{(0)} \otimes B_1 \otimes I_4) x^{(2)} + (\phi_1 \otimes I_4)(I_4 \otimes B_1 \otimes I_4^{(0)}) x^{(2)} \Big] + O^{[3+]} \\
&= B^T S_3^T \Big[(I_4 \otimes \phi_1)(B_1 \otimes I_4) x^{(2)} + (\phi_1 \otimes I_4)(I_4 \otimes B_1) x^{(2)} \Big] + O^{[3+]} \\
&= B^T S_3^T \left[B_1 \otimes \phi_1 + \phi_1 \otimes B_1 \right] x^{(2)} + O^{[3+]} \\
&= B^T S_3^T \tilde{B}_3 x^{(2)} + O^{[3+]}
\end{aligned} \tag{4.21}
$$

and

$$
\begin{aligned}
B^T \bar{S}_3(\alpha) S_3 B &= B^T S_3^T \bar{B}_3 \alpha \\
B^T \bar{S}_3(\alpha) S_3 B_1 x &= x^T B_1^T S_3^T \bar{B}_3 \alpha = row(B_1^T S_3^T \bar{B}_3 \phi_1) x^{(2)} + O^{[3+]}
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{B}_4 &= (I_4^{(0)} \otimes B \otimes I_4^{(2)}) + (I_4^{(1)} \otimes B \otimes I_4^{(1)}) + (I_4^{(2)} \otimes B \otimes I_4^{(0)}) \\
\tilde{B}_3 &= B_1 \otimes \phi_1 + \phi_1 \otimes B_1 \\
\bar{B}_3 &= (I_4^{(0)} \otimes B \otimes I_4^{(1)}) + (I_4^{(1)} \otimes B \otimes I_4^{(0)}).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
E^T \bar{S}_4(\alpha) S_4 B &= B^T S_4^T \bar{E}_4 \alpha^{(2)}, \\
x^T E_1^T \bar{S}_3(\alpha) S_3 B &= B^T S_3^T \tilde{E}_3 x^{(2)} + O^{[3+]} \\
E^T \bar{S}_3(\alpha) S_3 B &= B^T S_3^T \bar{E}_3 \alpha^{(1)} \\
E^T \bar{S}_3(\alpha) S_3 B_1 x &= row(B_1^T S_3^T \bar{E}_3 \phi_1) x^{(2)} + O^{[3+]}
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{E}_4 &= (I_4^{(0)} \otimes E \otimes I_4^{(2)}) + (I_4^{(1)} \otimes E \otimes I_4^{(1)}) + (I_4^{(2)} \otimes E \otimes I_4^{(0)}) \\
\tilde{E}_3 &= E_1 \otimes \phi_1 + \phi_1 \otimes E_1 \\
\bar{E}_3 &= (I_4^{(0)} \otimes E \otimes I_4^{(1)}) + (I_4^{(1)} \otimes E \otimes I_4^{(0)}).
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
H_{uu}(x,u^*(x),w^*(x)) &= (B^T + x^T B_1^T + x^{(2)^T} B_2^T + O^{[3+]}(x))\frac{\partial^2 V}{\partial \alpha^2}(B + B_1 x + B_2 x^{(2)} + O^{[3+]}(x)) \\
&\quad + D^T D \\
&= (B^T + x^T B_1^T + x^{(2)^T} B_2^T + O^{[3+]}(x))(P + \bar{S}_3(\alpha) S_3 + \bar{S}_4(\alpha) S_4) \\
&\quad \cdot (B + B_1 x + B_2 x^{(2)} + O^{[3+]}(x)) + D^T D \\
&= B^T P B + D^T D + B^T P B_1 x + B^T P B_2 x^{(2)} + x^T B_1^T P B + x^T B_1^T P B_1 x \\
&\quad + x^{(2)^T} B_2^T P B + B^T \bar{S}_3(\alpha) S_3 B + B^T \bar{S}_3(\alpha) S_3 B_1 x + x^T B_1^T \bar{S}_3(\alpha) S_3 B \\
&\quad + B^T \bar{S}_4(\alpha) S_4 B + O^{[3+]}(x) \\
&= B^T P B + D^T D + 2B^T P B_1 x + 2B^T P B_2 x^{(2)} + row(B_1^T P B_1) x^{(2)} \\
&\quad + B^T S_3^T \bar{B}_3 \alpha + row(B_1^T S_3^T \bar{B}_3 \phi_1) x^{(2)} + B^T S_3^T \tilde{B}_3 x^{(2)} \\
&\quad + B^T S_4^T \bar{B}_4 \alpha^{(2)} + O^{[3+]}(x) \\
&= B^T P B + D^T D + (2B^T P B_1 + B^T S_3^T \bar{B}_3 \phi_1) x \\
&\quad + [2B^T P B_2 + row(B_1^T P B_1) + B^T S_3^T \bar{B}_3 \phi_2 + row(B_1^T S_3^T \bar{B}_3 \phi_1) \\
&\quad + B^T S_3^T \tilde{B}_3 + B^T S_4^T \bar{B} \phi_1^{(2)}] x^{(2)} + O^{[3+]}(x) \\
&= B^T P B + D^T D + \hat{B}_1 x + \hat{B}_2 x^{(2)} + O^{[3+]}(x)
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{B}_1 &= 2B^T P B_1 + B^T S_3^T \bar{B}_3 \phi_1 \\
\hat{B}_2 &= 2B^T P B_2 + row(B_1^T P B_1) + B^T S_3^T \bar{B}_3 \phi_2 + row(B_1^T S_3^T \bar{B}_3 \phi_1) + B^T S_3^T \tilde{B}_3 + B^T S_4^T \bar{B}_4 \phi_1^{(2)}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
H_{wu}(x,u^*(x),w^*(x)) &= (E^T + x^T E_1^T + x^{(2)^T} E_2^T + O^{[3+]}(x))\frac{\partial^2 V}{\partial \alpha^2}(B + B_1 x + B_2 x^{(2)} + O^{[3+]}(x)) \\
&\quad + F^T D \\
&= E^T P B + F^T D + \hat{E}_1 x + \hat{E}_2 x^{(2)} + O^{[3+]}(x),
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{E}_1 &= B^T P E_1 + E^T P B_1 + B^T S_3^T \bar{E}_3 \phi_1 \\
\hat{E}_2 &= E^T P B_2 + B^T P E_2 + row(E_1^T P B_1) + B^T S_3^T \bar{B}_3 \phi_2 + row(B_1^T S_3^T \bar{E}_3 \phi_1) \\
&\quad + B^T S_3^T \tilde{E}_3 + B^T S_4^T \bar{E}_4 \phi_1^{(2)}.
\end{aligned}
$$

Finally noting that

$$\begin{aligned}
& H_{uu}^{-1}(x,u^*(x),w^*(x))H_{wu}(x,u^*(x),w^*(x)) \\
= & \left(B^TPB + D^TD + \hat{B}_1x + \hat{B}_2x^{(2)}\right)^{-1}\left(E^TPB + F^TD + \hat{E}_1x + \hat{E}_2x^{(2)}\right) + O^{[3+]}(x) \\
= & \frac{1}{B^TPB + D^TD}\left(1 - \frac{\hat{B}_1x + \hat{B}_2x^{(2)}}{B^TPB + D^TD}\right)\left(E^TPB + F^TD + \hat{E}_1x + \hat{E}_2x^{(2)}\right) + O^{[3+]}(x) \\
= & \frac{E^TPB + F^TD}{B^TPB + D^TD} + \left(\frac{\hat{E}_1}{B^TPB + D^TD} - \frac{(E^TPB + F^TD)\hat{B}_1}{(B^TPB + D^TD)^2}\right)x \\
& - \left(\frac{(E^TPB + F^TD)\hat{B}_2}{(B^TPB + D^TD)^2} - \frac{\hat{E}_2}{B^TPB + D^TD} + \frac{row(\hat{B}_1^T\hat{E}_1)}{(B^TPB + D^TD)^2}\right)x^{(2)} + O^{[3+]}(x)
\end{aligned}$$

gives the third order controller as follows

$$\begin{aligned}
u^{[3]} = & U_1x + U_2x^{[2]} + U_3x^{[3]} - \frac{E^TPB + F^TD}{B^TPB + D^TD}\left(w - W_1x - W_2x^{[2]} - W_3x^{[3]}\right) \\
& - \left(\frac{\hat{E}_1}{B^TPB + D^TD} - \frac{(E^TPB + F^TD)\hat{B}_1}{(B^TPB + D^TD)^2}\right)x\left(w - W_1x - W_2x^{[2]}\right) \\
& + \left(\frac{(E^TPB + F^TD)\hat{B}_2}{(B^TPB + D^TD)^2} - \frac{\hat{E}_2}{B^TPB + D^TD} + \frac{row(\hat{B}_1^T\hat{E}_1)}{(B^TPB + D^TD)^2}\right)x^{(2)}(w - W_1x)
\end{aligned} \tag{4.22}$$

Dropping the third order terms and the second order terms in (4.22) gives the second order controller as well as the linear controller as follows

$$\begin{aligned}
u^{[2]} = & U_1x + U_2x^{[2]} - \frac{E^TPB + F^TD}{B^TPB + D^TD}\left(w - W_1x - W_2x^{[2]}\right) \\
& - \left(\frac{\hat{E}_1}{B^TPB + D^TD} - \frac{(E^TPB + F^TD)\hat{B}_1}{(B^TPB + D^TD)^2}\right)x\,(w - W_1x) \\
u = & U_1x - (B^TPB + D^TD)^{-1}(E^TPB + F^TD)(w - W_1x).
\end{aligned} \tag{4.23}$$

4.3 Disturbance Attenuation of RTAC System

In this section, we will consider the disturbance attenuation problem for the discretized model of RTAC system. The continuous-time model is given in equation (1.15) of Chapter 1. Discretizing (1.15) via Euler's method with h as the sampling period gives the discrete-time state space equation of RTAC system as follows.

$$x(t+1) = A(x(t)) + B(x(t))u(t) + E(x(t))w(t),\ t = 0, 1, 2, \cdots, \tag{4.24}$$

where $x(t)$, $u(t)$ and $w(t)$ are the short-hand notations for $x(th)$, $u(th)$ and $w(th)$ with $t = 0, 1, 2, \cdots$, and

$$A(x) = hf(x) + x,\ B(x) = hg_2(x),\ E(x) = hg_1(x) \tag{4.25}$$

Our objective is find a full information approximate nonlinear H_∞ control law to achieve disturbance attenuation for the discrete-time RTAC system. For this purpose, let us introduce a performance output variable z defined as follows

$$z = C(x) + D(x)u + F(x)w \tag{4.26}$$

where $C(x) = x_1 + \delta(x_2 + x_3 + x_3)$ with δ a nonnegative real number, $D(x) = d$ with d a nonnegative real number, and $F(x) = 0$. When $\delta = d = 0$, the performance output $z = x_1$ is just the cart position, and when $\delta > 0$ and $d > 0$, the performance output z is a linear combination of the state variable x and the control input u. It is expected that a nonzero δ and/or nonzero d can be used to trade-off the performance and the required control effort.

To be specific, we will design a third order controller in this section. For this purpose, we need to expand the functions $A(x)$ $B(x)$, $E(x)$, $C(x)$, $D(x)$ and $F(x)$ up to third order. Doing so leads to the following matrices with the notations defined in (4.3).

$$\begin{aligned}
A &= \begin{bmatrix} 1 & h & 0 & 0 \\ -\frac{25}{24}h & 1 & 0 & 0 \\ 0 & 0 & 1 & h \\ \frac{5}{24}h & 0 & 0 & 1 \end{bmatrix},\ A_2 = 0_{4\times16},\ A_3 = \begin{bmatrix} (2,11) & (2,48) & (4,11) & (4,48) \\ h\frac{25}{576} & h\frac{5}{24} & -h\frac{65}{576} & -h\frac{1}{24} \end{bmatrix}_{4\times64}, \\
B &= \begin{bmatrix} 0 \\ -h\frac{5}{24} \\ 0 \\ h\frac{25}{24} \end{bmatrix},\ B_1 = 0_{4\times4},\ B_2 = \begin{bmatrix} (2,11) & (4,11) \\ h\frac{65}{576} & -h\frac{25}{576} \end{bmatrix}_{4\times16}, \\
E &= \begin{bmatrix} 0 \\ h\frac{25}{24} \\ 0 \\ -h\frac{5}{24} \end{bmatrix},\ E_1 = 0_{4\times4},\ E_2 = \begin{bmatrix} (2,11) & (4,11) \\ -h\frac{25}{576} & -h\frac{65}{576} \end{bmatrix}_{4\times16}, \\
C &= \begin{bmatrix} 1 & \delta & \delta & \delta \end{bmatrix},\ C_2 = 0_{1\times16}, C_3 = 0_{1\times64}, \\
D &= d,\ D_1 = 0_{1\times4},\ D_2 = 0_{1\times16}, \\
F &= 0,\ F_1 = 0_{1\times4},\ F_2 = 0_{1\times16}.
\end{aligned}$$

Let us first find the linear term of the solution of the DHJI equation with $\delta = 0.2$, $d = 1$, $h = 0.01$ and $\gamma = 10$ which can be straightforwardly obtained by solving (4.6) and (4.4) and the results are as follows

$$P = \begin{bmatrix} 402.17 & 27.642 & -22.978 & -37.215 \\ 27.642 & 453.29 & 8.7132 & -17.22 \\ -22.978 & 8.7132 & 1.5475 & 1.792 \\ -37.215 & -17.22 & 1.792 & 3.92 \end{bmatrix}, \tag{4.27}$$

and

$$\begin{aligned} U_1 &= \begin{bmatrix} -0.5648 & 0.92584 & -0.19988 & -0.27587 \end{bmatrix}, \\ W_1 &= \begin{bmatrix} 0.0032238 & 0.047527 & 0.00089446 & -0.0018349 \end{bmatrix}. \end{aligned}$$

Since $A_2 = 0$, $B_1 = 0$, $D_1 = 0$,$C_2 = 0$, $E_1 = 0$, $F_1 = 0$, it can be easily seen that the second order term of the solution of the DHJI equation is as follows

$$U_2 = 0_{1\times 10},\ W_2 = 0_{1\times 10},\ P_3 = 0_{1\times 20},\ S_3 = 0_{16\times 4}. \tag{4.28}$$

To obtain the third order term of the solution of the DHJI equation. let us first form L_4 and H_4 as follows.

$$\begin{aligned} L_4 &= M_4\phi_1^{(4)}N_4 - I_4^{(4)} \\ &\quad + \left[(\phi_1^T PB + \psi_1^T D)^T \otimes \xi_3^u M_3 + (\phi_1^T PE + \psi_1^T F - \gamma^2 W_1^T)^T \otimes \xi_3^w M_3\right] N_4, \end{aligned}$$

where

$$\begin{aligned} \xi_3^u &= M_4(r_{11}B_4^3 + r_{12}E_4^3)N_3, \\ \xi_3^w &= M_4(r_{21}B_4^3 + r_{22}E_4^3)N_3, \\ B_4^3 &= \sum_{\substack{t+s=3 \\ s\geq 0,\ t\geq 3}} B_4^{ts} = B_4^{30} = \sum_{i=1}^{4}\left(\sum_{\substack{l+m=3 \\ l\geq i-1,\ m\geq 4-i}} \Gamma_{l(i-1)} \otimes B_0 \otimes \Gamma_{m(4-i)}\right) \\ &= \Gamma_{00}\otimes B_0\otimes\Gamma_{33} + \Gamma_{11}\otimes B_0\otimes\Gamma_{22} + \Gamma_{22}\otimes B_0\otimes\Gamma_{11} + \Gamma_{33}\otimes B_0\otimes\Gamma_{00}, \\ E_4^3 &= \sum_{\substack{t+s=3 \\ s\geq 0,\ t\geq 3}} E_4^{ts} = E_4^{30} = \sum_{i=1}^{4}\left(\sum_{\substack{l+m=3 \\ l\geq i-1,\ m\geq 4-i}} \Gamma_{l(i-1)} \otimes E_0 \otimes \Gamma_{m(4-i)}\right) \\ &= \Gamma_{00}\otimes E_0\otimes\Gamma_{33} + \Gamma_{11}\otimes E_0\otimes\Gamma_{22} + \Gamma_{22}\otimes E_0\otimes\Gamma_{11} + \Gamma_{33}\otimes E_0\otimes\Gamma_{00}, \\ \Gamma_{00} &= 1,\ \Gamma_{11} = \phi_1,\ \Gamma_{22} = \phi_1\otimes\phi_1,\ \Gamma_{33} = \phi_1\otimes\phi_1\otimes\phi_1, \\ B_0 &= B,\ E_0 = E, \\ \phi_1 &= BU_1M_1 + EW_1M_1 + A. \\ \psi_1 &= DU_1M_1 + FW_1M_1 + C. \end{aligned}$$

The row vector H_4 can be calculated as follows:

$$
\begin{aligned}
H_4 &= -\frac{1}{2} row(\phi_2^T P \phi_2 + \psi_2^T \psi_2 - \gamma^2 M_2^T W_2^T W_2 M_2) N_4 - P_3 M_3 \Gamma_{43} N_4 \\
&\quad - row \left(\begin{bmatrix} P\phi_1 \\ \psi_1 \\ -\gamma^2 W_1 \end{bmatrix}^T \begin{bmatrix} (B\eta_3^u + E\eta_3^w) M_3 + A_3 + B_3^u + E_3^w \\ (D\eta_3^u + F\eta_3^w) M_3 + C_3 + D_3^u + F_3^w \\ \eta_3^w M_3 \end{bmatrix} \right) N_4,
\end{aligned}
$$

where,

$$
\begin{aligned}
\psi_2 &= DU_2 M_2 + FW_2 M_2 + (C_2 + D_2^u + F_2^w), \\
\Gamma_{43} &= \phi_1 \otimes \phi_1 \otimes \phi_2 + \phi_2 \otimes \phi_1 \otimes \phi_1 + \phi_1 \otimes \phi_2 \otimes \phi_1, \\
\begin{bmatrix} \eta_3^u \\ \eta_3^w \end{bmatrix} &= -R^{-1} \begin{bmatrix} G_3^u \\ G_3^w \end{bmatrix} N_3, \\
G_3^u &= B^T P (A_3 + B_3^u + E_3^w) + D^T (C_3 + D_3^u + F_3^w) \\
&\quad + row(B_1^T P \phi_2 + D_1^T \psi_2) + row(B_2^T P \phi_1 + D_2^T \psi_1) + P_3 M_3 B_3^3, \\
G_3^w &= E^T P (A_3 + B_3^u + E_3^w) + F^T (C_3 + D_3^u + F_3^w) \\
&\quad + row(E_1^T P \phi_2 + F_1^T \psi_2) + row(E_2^T P \phi_1 + F_2^T \psi_1) + P_3 M_3 E_3^3. \\
B_2^u &= B_1 (I_4 \otimes U_1 M_1), \\
E_2^w &= E_1 (I_4 \otimes W_1 M_1), \\
D_2^u &= D_1 (I_4 \otimes U_1 M_1), \\
F_2^w &= F_1 (I_4 \otimes W_1 M_1), \\
B_3^u &= B_1 (I_4 \otimes U_2 M_2) + B_2 (I_4^{(2)} \otimes U_1 M_1), \\
E_3^u &= E_1 (I_4 \otimes W_2 M_2) + E_2 (I_4^{(2)} \otimes W_1 M_1), \\
D_3^u &= D_1 (I_4 \otimes U_2 M_2) + D_2 (I_4^{(2)} \otimes U_1 M_1), \\
F_3^u &= F_1 (I_4 \otimes W_2 M_2) + F_2 (I_4^{(2)} \otimes W_1 M_1),
\end{aligned}
$$

$$
\begin{aligned}
B_3^3 &= \sum_{\substack{t+s=3\\ s\geq 0,\ t\geq 2}} B_3^{ts} = B_3^{30} + B_3^{21} \\
&= \sum_{i=1}^{3}\left(\sum_{\substack{l+m=3\\ l\geq i-1,\ m\geq 3-i}} \Gamma_{l(i-1)} \otimes B \otimes \Gamma_{m(3-i)} \right) + \sum_{i=1}^{3}\left(\sum_{\substack{l+m=2\\ l\geq i-1,\ m\geq 3-i}} \Gamma_{l(i-1)} \otimes B_1 \otimes \Gamma_{m(3-i)} \right) \\
&= \sum_{\substack{l+m=3\\ l\geq 0,\ m\geq 2}} (\Gamma_{l0} \otimes B \otimes \Gamma_{m2}) + \sum_{\substack{l+m=3\\ l\geq 1,\ m\geq 1}} (\Gamma_{l1} \otimes B \otimes \Gamma_{m1}) + \sum_{\substack{l+m=3\\ l\geq 2,\ m\geq 0}} (\Gamma_{l2} \otimes B \otimes \Gamma_{m0}) \\
&\quad + \sum_{\substack{l+m=2\\ l\geq 0,\ m\geq 2}} (\Gamma_{l0} \otimes B_1 \otimes \Gamma_{m2}) + \sum_{\substack{l+m=2\\ l\geq 1,\ m\geq 1}} (\Gamma_{l1} \otimes B_1 \otimes \Gamma_{m1}) + \sum_{\substack{l+m=2\\ l\geq 2,\ m\geq 0}} (\Gamma_{l2} \otimes B_1 \otimes \Gamma_{m0}) \\
&= \Gamma_{00} \otimes B \otimes \Gamma_{32} + \Gamma_{10} \otimes B \otimes \Gamma_{22} + \Gamma_{11} \otimes B \otimes \Gamma_{21} + \Gamma_{21} \otimes B \otimes \Gamma_{11} \\
&\quad + \Gamma_{32} \otimes B \otimes \Gamma_{00} + \Gamma_{22} \otimes B \otimes \Gamma_{10} + \Gamma_{00} \otimes B_1 \otimes \Gamma_{22} + \Gamma_{11} \otimes B_1 \otimes \Gamma_{11} + \Gamma_{22} \otimes B_1 \otimes \Gamma_{00}, \\
E_3^3 &= \sum_{\substack{t+s=3\\ s\geq 0,\ t\geq 2}} E_3^{ts} = E_3^{30} + E_3^{21} \\
&= \sum_{i=1}^{3}\left(\sum_{\substack{l+m=3\\ l\geq i-1,\ m\geq 3-i}} \Gamma_{l(i-1)} \otimes E \otimes \Gamma_{m(3-i)} \right) + \sum_{i=1}^{3}\left(\sum_{\substack{l+m=2\\ l\geq i-1,\ m\geq 3-i}} \Gamma_{l(i-1)} \otimes E_1 \otimes \Gamma_{m(3-i)} \right) \\
&= \Gamma_{00} \otimes E \otimes \Gamma_{32} + \Gamma_{10} \otimes E \otimes \Gamma_{22} + \Gamma_{11} \otimes E \otimes \Gamma_{21} + \Gamma_{21} \otimes E \otimes \Gamma_{11} \\
&\quad + \Gamma_{32} \otimes E \otimes \Gamma_{00} + \Gamma_{22} \otimes E \otimes \Gamma_{10} + \Gamma_{00} \otimes E_1 \otimes \Gamma_{22} + \Gamma_{11} \otimes E_1 \otimes \Gamma_{11} + \Gamma_{22} \otimes E_1 \otimes \Gamma_{00},
\end{aligned}
$$

and

$$
\begin{aligned}
\Gamma_{32} &= \phi_2 \otimes \phi_1 + \phi_1 \otimes \phi_2,\ \Gamma_{10} = 0_{1\times 4},\ \Gamma_{21} = \phi_2, \\
\phi_2 &= BU_2M_2 + EW_2M_2 + (A_2 + B_2^u + E_2^w).
\end{aligned}
$$

Using (4.8) and (4.13) with $k = 4$ gives

$$
U_3 = P_4\xi_3^u + \eta_3^u,\ W_3 = P_4\xi_3^w + \eta_3^w,
$$

and then here

$$
\begin{array}{llllll}
P_4 = H_4L_4^{-1} = [178.42 & 83.024 & -8.9588 & -194.29 & 275.14 \\
\quad 75.947 & 28.702 & -13.255 & -69.447 & 29.016 \\
\quad 98.195 & -31.454 & -183.28 & 5.2063 & -73.57 \\
\quad -126.36 & -1.1299 & 41.294 & 71.673 & 28.639 \\
\quad 95.619 & 47.355 & -46.764 & 21.619 & 26.429 \\
\quad 42.78 & -9.333 & -49.1 & 32.38 & 50.18 \\
\quad -0.097963 & -2.7888 & -4.9062 & -7.3569 & -3.9893]_{1\times 35},
\end{array}
$$

$$
\begin{array}{lllll}
U_3 = [2.1846 & 0.85148 & 0.88087 & -0.54069 & 2.5123 \\
\quad 0.63429 & 1.8631 & -0.46251 & -1.6363 & -1.1538 \\
\quad 1.2889 & 0.025736 & -1.1789 & 0.080127 & -0.55424 \\
\quad -1.3863 & 0.00027142 & 0.019299 & 0.29456 & 0.26984]_{1\times 20},
\end{array}
$$

$$
\begin{array}{lllll}
W_3 = [0.011915 & 0.056461 & 0.0093656 & 0.0023817 & 0.033882 \\
\quad -0.0051857 & -0.03302 & -0.00080421 & -0.010459 & -0.014978 \\
\quad 0.040698 & 0.014287 & -0.01618 & 0.0033511 & 0.0042611 \\
\quad 0.0059355 & -0.00093166 & -0.0048367 & 0.0037494 & 0.0055233]_{1\times 20}.
\end{array}
$$

Applying (4.23) and (4.22), we have the third order controller as follows

$$
\begin{aligned}
u^{[3]} &= U_1x + U_3x^{[3]} - \frac{E^TPB}{B^TPB + D^TD}\left(w - W_1x - W_3x^{[3]}\right) \\
&+ \left(\frac{E^TPB\tilde{B}}{(B^TPB + D^TD)^2} - \frac{\tilde{E}}{B^TPB + D^TD}\right)x^{(2)}(w - W_1x),
\end{aligned}
$$

and the linear controller as follows

$$
\begin{aligned}
u &= U_1x - (B^TPB + D^TD)^{-1}(E^TPB + F^TD)(w - W_1x) \\
&= U_1x - (B^TPB + D^TD)^{-1}(E^TPB)(w - W_1x),
\end{aligned}
$$

4.4 Computer Simulation

Here we will present some computer simulation results for the discretized RTAC system. These computer simulations have been conducted to evaluate the performance of our control law.

First, we consider the stabilizing properties of controllers. With initial condition $x(0) = [\ 1 \ \ -1 \ \ 1 \ \ -1\ ]$, Figure 4.1 shows the system behaviors under different controllers without disturbances. both linear controller and third-order controller are able to stabilize the closed-loop system.

Frequency	Nonlinear controller	Linear controller
$\omega = 2$	1.0335	1.1543
$\omega = 3$	0.43105	0.44706
$\omega = 4$	0.23858	0.24053

Table 4.1: Maximal steady state amplitudes of x_1 with $A_m = 3.5$.

Next, we show the phase portraits of the states to compare the controllers with respect to the attraction region for the closed-loop system. For initial condition $x(0) = [\ 1.5 \quad -1.5 \quad 1.5 \quad -1.5\ ]$, Figure 4.2 tells the third-order controller performs better than the linear one, since the state trajectories of closed-loop system under the third-order controller can approach to the origin but the trajectories under the linear controller go to a limit cycle.

Finally, we discuss the disturbance attenuation properties of the two controllers. Figure 4.3 shows the time responses of the four states of the open-loop system under a sinusoidal disturbance $F(t) = sin(2.5t)$ with zero initial condition. Part (a) and (b) of Figure 4.4 show the time responses of the position output x_1 of the closed-loop system and the control input under the linear and third-order controller, respectively, with the disturbance $F(t) = sin(2.5t)$ and zero initial condition. Part (a) and (b) of Figure 4.5 repeat the scenario of Figure 4.4 but with the disturbance $F(t) = 5sin(2.5t)$.

To demonstrate more scenarios, Table 1 lists the steady state amplitudes of the time response of the position variable x_1 for several different frequencies with $A_m = 3.5$ and Table 2 lists the steady state amplitudes of the time response of the position variable x_1 for several different amplitudes with $\omega = 2.0$. It can be seen that when the amplitude of the disturbance is relatively small, the performances of the linear and third-order controllers are similar. But as the amplitude of the disturbance increases, the performance of the third-order controller is better than the linear controller. In particular, for the case where $\omega = 2.5$, when the amplitude of the disturbance is increased to 5, the trajectories of the closed-loop system resulting from the linear controller go unbounded while the trajectories of the closed-loop system resulting from the third-order controller remain bounded. This case shows that the stability domain of the closed-loop system resulting from the third-order controller may be larger than that of the closed-loop system resulting from the linear controller.

The programs of the simulation is attached in Appendix.

Amplitude	Nonlinear controller	Linear controller
$A_m = 0.5$	`0.16455`	`0.16485`
$A_m = 2$	`0.61135`	`0.65917`
$A_m = 3.5$	`1.0335`	`1.1543`

Table 4.2: Maximal steady state amplitudes of x_1 with $\omega = 2$.

4.5 Conclusions

In this section, we have applied an approximation method to design a discrete nonlinear H_∞ controller for the discretized RTAC system. The outcome of this study shows, like the continuous-time case, on one hand that the complex algorithm developed in [23] can be successfully implemented by routine software package, and on the other hand shows that, a higher order approximation of nonlinear H_∞control law may perform better than a linear H_∞ controller, a conclusion that may not be taken for granted.

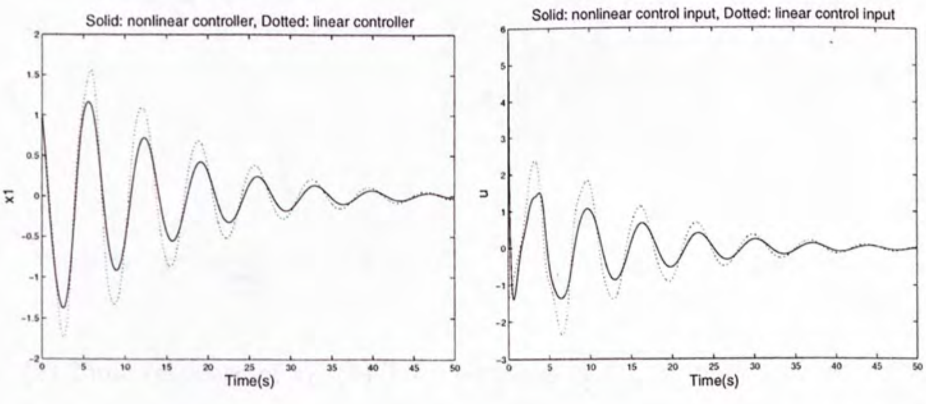

Figure 4.1: Time responses of x_1 and u for the linear controller and the third order controller without disturbances.

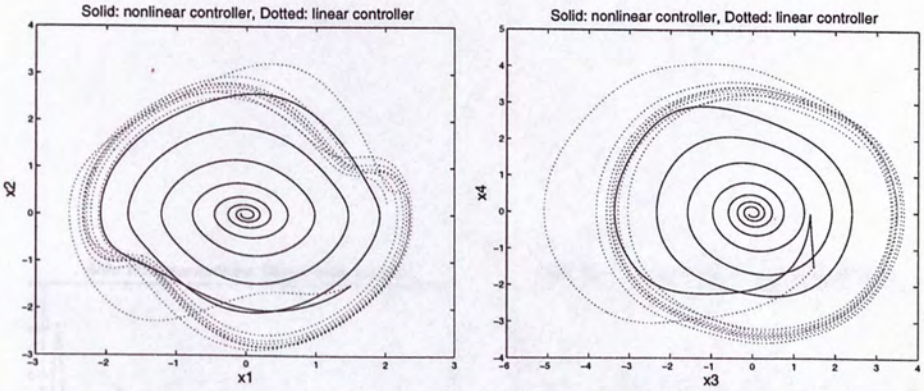

Figure 4.2: Phase portrait of ξ for the linear controller, the third order controller and the fifth order controller.

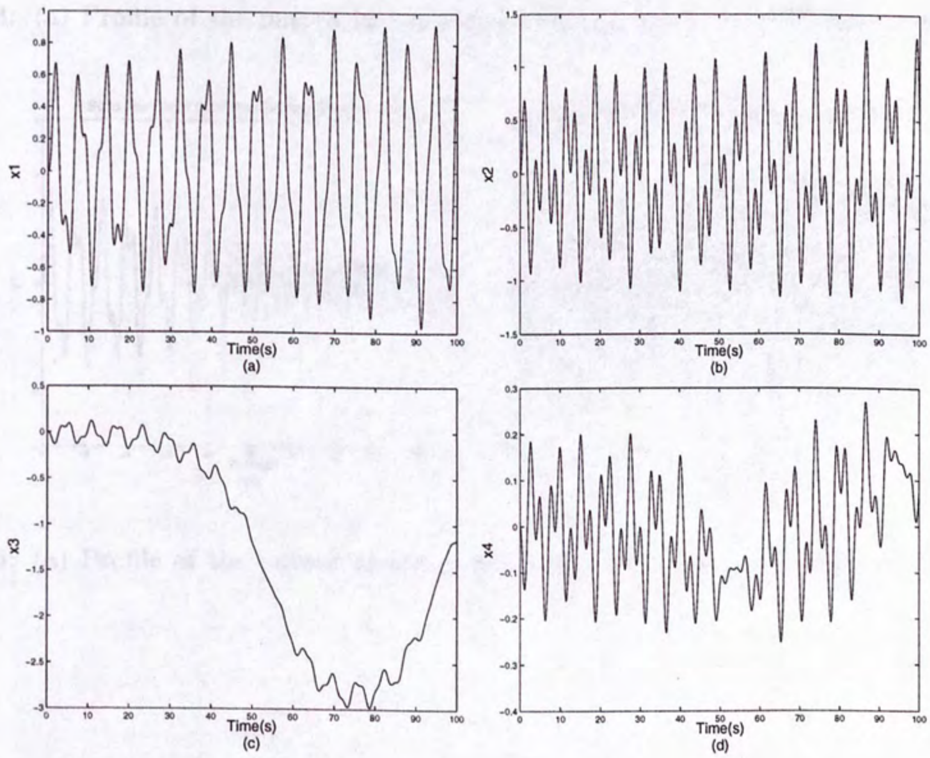

Figure 4.3: (a) Time response of x_1, (b) Time response of x_2, (c) Time response of x_3, and (d) Time response of x_4.

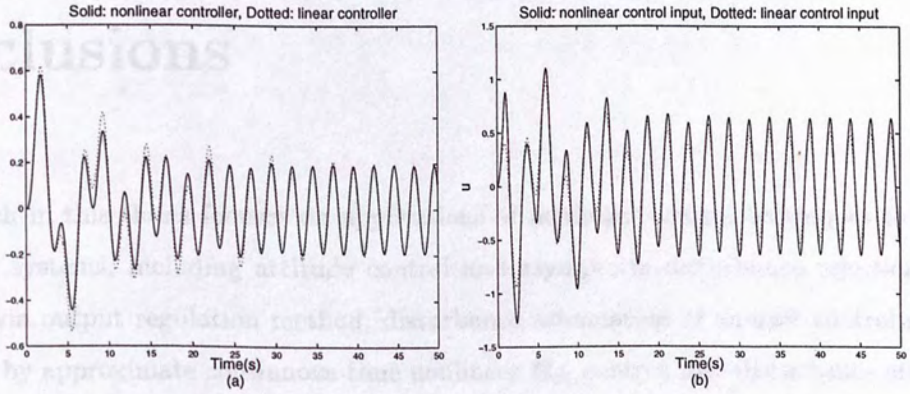

Figure 4.4: (a) Profile of the output of the closed-loop system, and (b) Profile of the control input u.

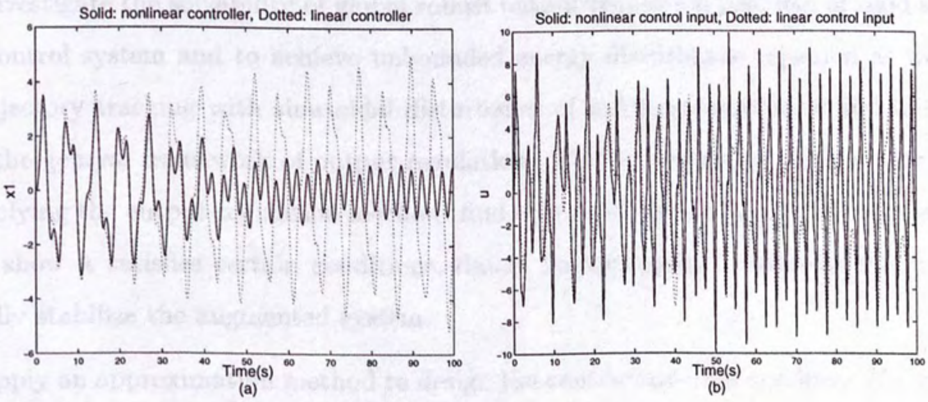

Figure 4.5: (a) Profile of the output of the closed-loop system, and (b) Profile of the control input u.

Chapter 5

Conclusions

The research in this thesis focuses on applications of nonlinear control techniques to practical engineering systems, including attitude control and asymptotic disturbance rejection of rigid spacecraft via output regulation method, disturbance attenuation of aircraft control system in windshears by approximate continuous-time nonlinear H_∞ control, and disturbance attenuation of the discrete nonlinear benchmark system by approximate discrete-time nonlinear H_∞ control.

The main contributions of the thesis are in order:

• To investigate the solvability of global robust output regulation problem of rigid spacecraft attitude control system and to achieve unbounded energy disturbance rejection as well as reference trajectory tracking with sinusoidal disturbance of unknown amplitude and initial phase based on the general framework of output regulation. We have overcame three major obstacles during applying the output regulation method: find a closed-form solution of the regulator equations and show it satisfies certain conditions, design an appropriate nonlinear internal model, and globally stabilize the augmented system.

• To apply an approximation method to design the continuous-time nonlinear H_∞ controllers for aircraft control systems in windshears. The performances of the computer simulations shows that an approximation of nonlinear H_∞ control law may perform better than a linear controller. Moreover, the procedure of programming proves the complex algorithm to achieve the approximation nonlinear controller can be easily realized by the routine software. It brings much convenience for engineering design.

• To apply an approximation method to design the discrete-time nonlinear H_∞ controllers for RTAC system. Similarly with continuous-time case, two contributions are achieved here. On one hand, the approximation of nonlinear controller may perform better than the linear controller. On the other hand, the programming realization of the complex algorithm is feasible.

Just as mentioned in Chapter 2, the controller design in attitude control of rigid spacecraft does not consider the degree of optimality of the controller. The natural next step is to improve the transient response of the closed-loop system and minimize energy cost of the controller. On the other hand, it is interesting to consider the application of output regulation to some underactuated systems, which sometimes comes from the actuator failures or various construction constraints.

My future work will also focus on the following problems:

• Continue to investigate the global robust output regulation problem and apply it to more emerging and significative engineering problems.

• Study more nonlinear control techniques in order to obtain more methods to design nonlinear control laws for practical engineering systems.

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Appendix A

Programs

Programs for flight control in windshears

```
%% Program to calculate the coefficients
%% File name: main_windshear.m
clear all
close all
% find out f(x) g_1(x) g_2(x) h_1(x) k_12(x)
% xdot=f(x)+g_1(x)w+g_2(x)u
% z=h_1(x)+k_12(x)u
% step 1
% get A A_2 ... A_n from f(x); get C_1 from h_1(x); B_1 from g_1; B_2 from
% g_2, R_2 from k_12 and gamma
% get the system parameters
a0=288375/8;
a1=425/8;
a2=-5/32;
b0=0.12;
b1=-0.716;
b2=4.1;
c0=0.5314;
c1=3.95;
g=32.1311;
rho=0.0022;
S=3500;
m=5602;
V0=288.2202;
```

```
delta=0.2617623;
M=2;

A=[ (a1+2*a2*V0-b0*rho*S*V0)/m -g;
    -delta*a0/(m*V0*V0)+a2*delta/m+c0*rho*S/(2*m)+g/V0^2 0];
A_2=zeros(2,4);
A_2(1,1)=a2/m-b0*rho*S/(2*m);
A_2(2,1)=delta*a0/(m*V0^3)-g/V0^3;
A_3=zeros(2,8);
A_3(2,1)=-delta*a0/(m*V0^4)+g/V0^4;

C_1=[(delta*a1+2*delta*a2*V0+c0*rho*S*V0)/m 0;
    0 0];
C_2=[(delta*a2+0.5*c0*rho*S)/m (a1+2*a2*V0-b0*rho*S*V0)/m 0 0;
    0 0 0 0];
C_3=[0 (a2-0.5*b0*rho*S)/m 0 0 0 0 0 0;
    0 0 0 0 0 0 0 0];

B_11=[1 0]';
B_12=[0 -1/V0]';
B_1=[B_11 B_12];
B1_11=[0 0;
    0 1/V0];
B1_12=[0 1;
    1/V0^2 0];
B2_11=zeros(2,4);
B2_11(2,2)=-1/V0^2;
B2_12=zeros(2,4);
B2_11(2,1)=-1/V0^3;

B_2=[-b1*rho*S*V0^2/(2*m) a0/(m*V0)+a1/m+a2*V0/m+c1*rho*S*V0/(2*m)]';
B_21=B_2;
B1_21=[-b1*rho*S*V0/m 0;
    -a0/(m*V0^2)+a2/m+c1*rho*S/(2*m) 0];
B2_21=zeros(2,4);
B2_21(1,1)=-b1*rho*S/(2*m);
B2_21(2,1)=a0/(m*V0^3);

R_2=M*M;
gamma=2;

% step 2
% get H_px H_xx H_pp
H_px=A;
```

```
H_xx=C_1'*C_1;
H_pp=B_1*B_1'/gamma^2-B_2*R_2^(-1)*B_2';

% step 3
% get P from the Riccati Equation
% using care function
m1 = size(B_1,2);
m2 = size(B_2,2);
R = [-gamma^2*eye(m1) zeros(m1,m2) ; zeros(m2,m1) R_2];
B=[B_1 B_2];
P=care(H_px,B,H_xx,R);

% step 4
% S_2=P
% get E_3 F_3 I2_3 I1_3
S_2=P;
E_3=m2v(S_2*A_2)';
F_2=m2v(C_1'*C_1)';
F_3=m2v(C_1'*C_2)'+m2v(C_2'*C_1)';
W2_11=m2v(S_2*B1_11)';
W2_12=m2v(S_2*B1_12)';
Y1_11=B_11'*S_2';
Y1_12=B_12'*S_2';
I1_3=m2v(W2_11'*Y1_11)'+m2v(W2_12'*Y1_12)';
W2_21=m2v(S_2*B1_21)';
Y1_21=B_21'*S_2';
I2_3=m2v(W2_21'*R_2^(-1)*Y1_21)';

% step 5
% get M_3 N_3 V_3 U_3 M_2 N_2
% get M_2 N_2
M_2=zeros(3,4);
N_2=zeros(4,3);
l=0;
k=0;
for i1=1:2
    for i2=1:2
        if i1>i2
            l=l;
        else
            l=l+1;
            FM2_2=(i1-1)*2+i2;
            M_2(l,FM2_2)=1;
            h=[i1 i2];
```

```
                for j1=1:2
                    for j2=1:2
                        if j1~=j2
                            k=k+1;
                            FN2_2(k)=(h(j1)-1)*2+h(j2);
                            N_2(FN2_2(k),l)=1;
                        end
                    end
                end
            end
        end
    end

M_3=zeros(4,8);
N_3=zeros(8,4);
l=0;
k=0;
for i1=1:2
    for i2=1:2
        for i3=1:2
                if i2>i3 | i1>i2
                    l=l;
                else
                    l=l+1;
                    FM2_3=(i1-1)*2^2+(i2-1)*2+i3;
                    M_3(l,FM2_3)=1;
                    h=[i1 i2 i3 ];
                    for j1=1:3
                        for j2=1:3
                            for j3=1:3
                                if j1~=j2 & j1~=j3 &   j2~=j3
                                    k=k+1;
                                    FN2_3(k)=(h(j1)-1)*2^2+(h(j2)-1)*2+h(j3);
                                    N_3(FN2_3(k),l)=1;
                                end
                            end
                        end
                    end
                end
        end
    end
end
```

```
V_3=-(E_3+(F_3-2*I2_3)/2+I1_3/gamma^2)*N_3;
U_3=M_3*(kron((H_px+H_pp*P),eye(4))+kron(kron(eye(2),(H_px+H_pp*P)),eye(2))
+kron(eye(4),(H_px+H_pp*P)))*N_3;

% step 6
% get P_3 and then S_3
P_3=V_3*U_3^(-1);
PM_3=P_3*M_3;
for i=1:2
    PP(i,:)=PM_3((i-1)*4+1:i*4);
    P1_3(i,:)=PP(i,:);
end
for i=1:4
    PPP(i,:)=PM_3((i-1)*2+1:i*2);
end
for j=1:2
    for k=1:2
        P2_3(j,(k-1)*2+1:k*2)=PPP((k-1)*2+j,:);
    end
end

for i=1:8
    PPPP(i)=PM_3(i);
end
for j=1:2
    for k=1:4
        P3_3(j,k)=PPPP((k-1)*2+j);
    end
end

S_3=(P3_3+P1_3+P2_3)';

% step 7
% for n=3
% from S_n to get E/F/Z/I1/I2/G1/G2_n+1
E_4=m2v(S_2*A_3)'+m2v(S_3*A_2)';
F_4=m2v(C_1'*C_3)'+m2v(C_2'*C_2)'+m2v(C_3'*C_1)';
Z_4=m2v(S_3*H_pp*S_3')';
Y2_11=B_11'*S_3';
Y2_12=B_12'*S_3';
W3_11=m2v(S_2*B2_11)'+m2v(S_3*B1_11)';
W3_12=m2v(S_2*B2_12)'+m2v(S_3*B1_12)';
```

```
I1_4=m2v(W2_11'*Y2_11)'+m2v(W3_11'*Y1_11)'+m2v(W2_12'*Y2_12)'
+m2v(W3_12'*Y1_12)';
Y2_21=B_21'*S_3';
W3_21=m2v(S_2*B2_21)'+m2v(S_3*B1_21)';
I2_4=m2v(W2_21'*R_2^(-1)*Y2_21)'+m2v(W3_21'*R_2^(-1)*Y1_21)';
G1_4=m2v(W2_11'*W2_11)'+m2v(W2_12'*W2_12)';
G2_4=m2v(W2_21'*R_2^(-1)*W2_21)';

% get V_n+1 U_n+1
M_4=zeros(5,16);
N_4=zeros(16,5);
l=0;
k=0;
for i1=1:2
    for i2=1:2
        for i3=1:2
            for i4=1:2
                if i3>i4 | i2>i3 | i1>i2
                    l=l;
                else
                    l=l+1;
                    FM2_4=(i1-1)*2^3+(i2-1)*2^2+(i3-1)*2+i4;
                    M_4(l,FM2_4)=1;
                    h=[i1 i2 i3 i4];
                    for j1=1:4
                        for j2=1:4
                            for j3=1:4
                                for j4=1:4
                                    if j1~=j2 & j1~=j3 & j1~=j4 & j2~=j3 & j2~=j4
                                    & j3~=j4
                                    k=k+1;
                                    FN2_4(k)=(h(j1)-1)*2^3+(h(j2)-1)*2^2
                                        +(h(j3)-1)*2+h(j4);
                                    N_4(FN2_4(k),l)=1;
                                    end
                                end
                            end
                        end
                    end
                end
            end
        end
    end
end
```

```
V_4=-0.5*(Z_4+2*E_4+F_4+(2*I1_4+G1_4)/gamma^2-2*I2_4-G2_4)*N_4;
U_4=M_4*(kron((H_px+H_pp*P),eye(8))+kron(kron(eye(2),(H_px+H_pp*P)),eye(4))
+kron(kron(eye(4),(H_px+H_pp*P)),eye(2))+kron(eye(8),(H_px+H_pp*P)))*N_4;

% get P_n+1
P_4=V_4*U_4^(-1);

% get S_n+1
PM_4=P_4*M_4;
for i=1:2
    TT(i,:)=PM_4((i-1)*8+1:i*8);
    P1_4(i,:)=TT(i,:);
end
for i=1:4
    TTT(i,:)=PM_4((i-1)*4+1:i*4);
end
for j=1:2
    for k=1:2
            P2_4(j,(k-1)*4+1:k*4)=TTT((k-1)*2+j,:);
    end
end

for i=1:8
    TTTT(i,:)=PM_4((i-1)*2+1:i*2);
end
for j=1:2
    for k=1:4
            P3_4(j,(k-1)*2+1:k*2)=TTTT((k-1)*2+j,:);
    end
end

for i=1:16
    TTTTT(i)=PM_4(i);
end
for j=1:2
    for k=1:8
            P4_4(j,k)=TTTTT((k-1)*2+j);
    end
end

S_4=(P4_4+P1_4+P2_4+P3_4)';

% get the controllers
```

```
u_1=-R_2^(-1)*B_2'*P;
u_2=-R_2^(-1)*(B_2'*S_3'+m2v(B1_21'*P)')*N_2;
u_3=-R_2^(-1)*(B_2'*S_4'+m2v(B1_21'*S_3')'+m2v(B2_21'*P)')*N_3;

% save the parameters which will be used in the rest programs
save('U','u_1','u_2','u_3','a0','a1','a2','b0','b1','b2','c0','c1','V0','delta','m','rho','g','S','M');

% end of the program


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%% Main Program to simulate the closed-loop system
%% File name: main_sys.m

close all
clear all

global u_1 u_2 u_3

% initial states and parameters
load('U');
x0=[-10 0.1]';
t0=0;tfinal=250
tspan=[t0 tfinal];

% ode to calculate the states
[tctrl1,xctrl1]=ode45('WD1',tspan,x0);
[tctrl2,xctrl2]=ode45('WD2',tspan,x0);
[tctrl3,xctrl3]=ode45('WD3',tspan,x0);

% calculate the controller and output from the states
[i,j]=size(tctrl1);
for j=1:i
    u1=u_1*[xctrl1(j,1);xctrl1(j,2)];
    uctrl1(j,:)=u1;
    h1(j,:)=(a0+a1*(V0+xctrl1(j,1))+a2*(V0+xctrl1(j,1))^2)*(xctrl1(j,2)+delta)/m
            -b0*rho*S*(xctrl1(j,1)+V0)^2*xctrl1(j,2)/(2*m)
            +c0*rho*S*(xctrl1(j,1)+V0)^2/(2*m)-g;
end
[i,j]=size(tctrl2);
for j=1:i
    u1=u_1*[xctrl2(j,1);xctrl2(j,2)];
    u2=u_2*[xctrl2(j,1)^2;xctrl2(j,1)*xctrl2(j,2);xctrl2(j,2)^2];
```

```
    uctrl2(:,j)=u1+u2;
    h2(j,:)=(a0+a1*(V0+xctrl2(j,1))+a2*(V0+xctrl2(j,1))^2)*(xctrl2(j,2)+delta)/m
           -b0*rho*S*(xctrl2(j,1)+V0)^2*xctrl2(j,2)/(2*m)
           +c0*rho*S*(xctrl2(j,1)+V0)^2/(2*m)-g;
end
[i,j]=size(tctrl3);
for j=1:i
    u1=u_1*[xctrl3(j,1);xctrl3(j,2)];
    u2=u_2*[xctrl3(j,1)^2;xctrl3(j,1)*xctrl3(j,2);xctrl3(j,2)^2];
    u3=u_3*[xctrl3(j,1)^3;xctrl3(j,1)^2*xctrl3(j,2);xctrl3(j,2)^2*xctrl3(j,1);xctrl3(j,2)^3];
    uctrl3(:,j)=u1+u2+u3;
    h3(j,:)=(a0+a1*(V0+xctrl3(j,1))+a2*(V0+xctrl3(j,1))^2)*(xctrl3(j,2)+delta)/m
           -b0*rho*S*(xctrl3(j,1)+V0)^2*xctrl3(j,2)/(2*m)
           +c0*rho*S*(xctrl3(j,1)+V0)^2/(2*m)-g;
end

% get the max steady state value of output
[i,j]=size(tctrl1);
j=i-200;
k1=h1(j,1);
for k=j:i
    if k1>h1(k,1)
        k1=k1;
    else k1=h1(k,1);
    end
end
[i,j]=size(tctrl2);
k2=h2(i-200,1);
for k=i-200:i
    if k2>h2(k,1)
        k2=k2;
    else k2=h2(k,1);
    end
end
[i,j]=size(tctrl3);
k3=h3(i-200,1);
for k=i-200:i
    if k3>h3(k,1)
        k3=k3;
    else k3=h3(k,1);
    end
end

% plot the figures
```

```
figure(1)
plot(tctrl1, xctrl1(:,1), 'b:',tctrl2, xctrl2(:,1), 'r-',tctrl3, xctrl3(:,1), 'g-.');
title(['Dotted: linear controller, Solid: 2nd order controller, Dashdot: 3rd order controller ']);
xlabel('time(s)');
ylabel('x1 (ft sec-1)');

figure(2)
plot(tctrl1, xctrl1(:,2), 'b:',tctrl2, xctrl2(:,2), 'r-',tctrl3, xctrl3(:,2), 'g-.');
title(['Dotted: linear controller, Solid: 2nd order controller, Dashdot: 3rd order controller ']);
xlabel('time(s)');
ylabel('x2 (rad)');

figure(3)
plot(tctrl1, uctrl1, 'b:',tctrl2, uctrl2, 'r-',tctrl3, uctrl3, 'g-.');
title(['Dotted: linear controller, Solid: 2nd order controller, Dashdot: 3rd order controller ']);
xlabel('time(s)');
ylabel('u');

figure(4)
plot(tctrl1, h1, 'b:',tctrl2, h2, 'r-',tctrl3, h3, 'g-.');
title(['Dotted: linear controller, Solid: 2nd order controller, Dashdot: 3rd order controller ']);
xlabel('time(s)');
ylabel('output');

figure(5)
plot(xctrl1(:,1), xctrl1(:,2), 'b:',xctrl2(:,1), xctrl2(:,2), 'r-',xctrl3(:,1), xctrl3(:,2), 'g-.');
title(['Dotted: linear controller, Solid: 2nd order controller, Dashdot: 3rd order controller ']);
xlabel('x1');
ylabel('x2');

% end of the program


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%% ODE Program 1
%% File name: WD1.m

function xdot=System(t,x)

% system with 1st order controller
global u_1 u_2 u_3

% load parameters
```

```
load('U');

% get disturbance
wx0=50;
wh0=30;
T0=60;
w1=-wx0*2*pi/T0*cos(40*pi*t/T0);
w2=-wh0*2*pi/T0*sin(40*pi*t/T0);
w=13*[w1;w2];

% design of controller
u1=u_1*[x(1);x(2)];
u=u1;

% system
xdot(1)=a0/m+a1*(V0+x(1))/m+a2*(V0+x(1))^2/m-(b0+b1*u)*rho*S*(V0+x(1))^2/(2*m)
        -g*x(2)-[1 x(2)]*w-(a0+a1*V0+a2*V0^2-0.5*b0*rho*S*V0^2)/m;
xdot(2)=(a0/(m*(V0+x(1)))+a1/m+a2*(V0+x(1))/m)*(u+delta)
        +(c0+c1*u)*rho*S*(V0+x(1))/(2*m)-g/(V0+x(1))+(1/(V0+x(1)))*[x(2)-1]*w
        -((a0+a1*V0+a2*V0^2)*delta+0.5*c0*rho*S*V0^2-m*g)/(m*V0);

t
xdot=xdot(:);

% end of the program


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%% ODE Program 2
%% File name: WD2.m

function xdot=System(t,x)

% system with 2nd order controller
global u_1 u_2 u_3

% load parameters
load('U');

% get disturbance
wx0=50;
wh0=30;
T0=60;
```

```
w1=-wx0*2*pi/T0*cos(40*pi*t/T0);
w2=-wh0*2*pi/T0*sin(40*pi*t/T0);
w=13*[w1;w2];

% design of controller
u1=u_1*[x(1);x(2)];
u2=u_2*[x(1)^2;x(1)*x(2);x(2)^2];
u=(u1+u2);


%system
xdot(1)=a0/m+a1*(V0+x(1))/m+a2*(V0+x(1))^2/m-(b0+b1*u)*rho*S*(V0+x(1))^2/(2*m)
        -g*x(2)-[1 x(2)]*w-(a0+a1*V0+a2*V0^2-0.5*b0*rho*S*V0^2)/m;
xdot(2)=(a0/(m*(V0+x(1)))+a1/m+a2*(V0+x(1))/m)*(u+delta)
       +(c0+c1*u)*rho*S*(V0+x(1))/(2*m)-g/(V0+x(1))+(1/(V0+x(1)))*[x(2)
       -1]*w-((a0+a1*V0+a2*V0^2)*delta+0.5*c0*rho*S*V0^2-m*g)/(m*V0);

t
xdot=xdot(:);


% end of the program



%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%% ODE Program 3
%% File name: WD3.m

function xdot=System(t,x)

% system with 3st order controller
global u_1 u_2 u_3

% load parameters
load('U');

% get disturbance
wx0=50;
wh0=30;
T0=60;
w1=-wx0*2*pi/T0*cos(40*pi*t/T0);
w2=-wh0*2*pi/T0*sin(40*pi*t/T0);
w=13*[w1;w2];
```

```
% design of controller
u1=u_1*[x(1);x(2)];
u2=u_2*[x(1)^2;x(1)*x(2);x(2)^2];
u3=u_3*[x(1)^3;x(1)^2*x(2);x(2)^2*x(1);x(2)^3];
u=u1+u2+u3;

%system
xdot(1)=a0/m+a1*(V0+x(1))/m+a2*(V0+x(1))^2/m-(b0+b1*u)*rho*S*(V0+x(1))^2/(2*m)
        -g*x(2)-[1 x(2)]*w-(a0+a1*V0+a2*V0^2-0.5*b0*rho*S*V0^2)/m;
xdot(2)=(a0/(m*(V0+x(1)))+a1/m+a2*(V0+x(1))/m)*(u+delta)
        +(c0+c1*u)*rho*S*(V0+x(1))/(2*m)-g/(V0+x(1))+(1/(V0+x(1)))*[x(2)
        -1]*w-((a0+a1*V0+a2*V0^2)*delta+0.5*c0*rho*S*V0^2-m*g)/(m*V0);

t
xdot=xdot(:);

% end of the program

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Programs for disturbance attenuation of discretized RTAC system

```
%% Program to calculate the coefficients
%% File name: main_3ord.m

clear all
close all

% discrete-time system
% x_(k+1)=A(x_k)+B(x_k)u_k+E(x_k)w_k
% z_k=C(x_k)+D(x_k)u_k+F(x_k)w_k

% step 1
% get A A_2 ... A_n from A(x);
% get B B_2 ... B_n from B(x);
% get E E_2 ... E_n from E(x);
% get C C_2 ... C_n from C(x);
% get D D_2 ... D_n from D(x);
% get F F_2 ... F_n from F(x);
h=0.01;
A=h*[ 0 1 0 0;
    -25/24 0 0 0;
    0 0 0 1;
    5/24 0 0 0]+eye(4);
B=h*[0 -5/24 0 25/24]';
E=h*[0 25/24 0 -5/24]';
C=[1 0.2 0.2 0.2];
D=1;
F=0;
A_2=zeros(4,16);
A_3=zeros(4,64);
A_3(2,11)=h*25/576;
A_3(2,48)=h*5/24;
A_3(4,11)=-h*65/576;
A_3(4,48)=-h*1/24;
C_2=zeros(1,16);
C_3=zeros(1,64);
E_1=zeros(4,4);
E_2=zeros(4,16);
E_2(2,11)=-h*25/576;
E_2(4,11)=h*65/576;
```

```
B_1=zeros(4,4);
B_2=zeros(4,16);
B_2(2,11)=h*65/576;
B_2(4,11)=-h*25/576;
D_1=zeros(1,4);
D_2=zeros(1,16);
D_3=zeros(1,64);
F_1=zeros(1,4);
F_2=zeros(1,16);
F_3=zeros(1,64);
gamma=10;
n=4;
% step 2
% get P U_1 W_1 R
PB = [B E];
S = [C'*D C'*F];
PR = [D'*D D'*F;
    F'*D F'*F-gamma^2];
P=dare(A,PB,C'*C,PR,S);
R=[B'*P*B+D'*D B'*P*E+D'*F;
    E'*P*B+F'*D E'*P*E+F'*F-gamma^2];
UW=-R^(-1)*[B'*P*A+D'*C;E'*P*A+F'*C];
U_1=UW(1,:);
W_1=UW(2,:);

% get M_2 N_2
M_2=zeros(10,16);
N_2=zeros(16,10);
l=0;
k=0;
for i1=1:4
    for i2=1:4
        if i1>i2
            l=l;
        else
            l=l+1;
            FM4_2=(i1-1)*4+i2;
            M_2(l,FM4_2)=1;
            h=[i1 i2];
            for j1=1:2
                for j2=1:2
                    if j1~=j2
                        k=k+1;
```

```
                        FN4_2(k)=(h(j1)-1)*4+h(j2);
                        N_2(FN4_2(k),l)=1;
                    end
                end
            end
        end
    end
end

% get M_3 N_3
M_3=zeros(20,64);
N_3=zeros(64,20);
l=0;
k=0;
for i1=1:4
    for i2=1:4
        for i3=1:4
            if i2>i3 | i1>i2
                l=l;
            else
                l=l+1;
                FM4_3=(i1-1)*4^2+(i2-1)*4+i3;
                M_3(l,FM4_3)=1;
                h=[i1 i2 i3 ];
                for j1=1:3
                    for j2=1:3
                        for j3=1:3
                            if j1~=j2 & j1~=j3 &   j2~=j3
                                k=k+1;
                                FN4_3(k)
                                 =(h(j1)-1)*4^2+(h(j2)-1)*4+h(j3);
                                N_3(FN4_3(k),l)=1;
                            end
                        end
                    end
                end
            end
        end
    end
end
```

```
% get M_4 N_4
M_4=zeros(35,256);
N_4=zeros(256,35);
l=0;
k=0;
for i1=1:4
    for i2=1:4
        for i3=1:4
            for i4=1:4
                if i3>i4 | i2>i3 | i1>i2
                    l=l;
                else
                    l=l+1;
                    FM4_4=(i1-1)*4^3+(i2-1)*4^2+(i3-1)*4+i4;
                    M_4(l,FM4_4)=1;
                    h=[i1 i2 i3 i4];
                    for j1=1:4
                        for j2=1:4
                            for j3=1:4
                                for j4=1:4
                                    if j1~=j2 & j1~=j3 & j1~=j4 & j2~=j3 & j2~=j4
                                        & j3~=j4
                                        k=k+1;
                                        FN4_4(k)=(h(j1)-1)*4^3+(h(j2)-1)*4^2
                                         +(h(j3)-1)*4+h(j4);
                                        N_4(FN4_4(k),l)=1;
                                    end
                                end
                            end
                        end
                    end
                end
            end
        end
    end
end

% step 3
% get L_3 and H_3
M_1=eye(4);
phi_1=B*U_1*M_1+E*W_1*M_1+A;
psi_1=D*U_1*M_1+F*W_1*M_1+C;
RI=-R^(-1);
Gamma_00=1;
```

```
Gamma_11=phi_1;
Gamma_22=kron(phi_1,phi_1);
B2_3=kron(kron(Gamma_00,B),Gamma_22)+kron(kron(Gamma_11,B),Gamma_11)
     +kron( kron(Gamma_22,B),Gamma_00);
E2_3=kron(kron(Gamma_00,E),Gamma_22)+kron(kron(Gamma_11,E),Gamma_11)
     +kron(kron(Gamma_22,E),Gamma_00);
xiu_2=M_3*(RI(1,1)*B2_3+RI(1,2)*E2_3)*N_2;
xiw_2=M_3*(RI(2,1)*B2_3+RI(2,2)*E2_3)*N_2;
L_3=M_3*kron(kron(phi_1,phi_1),phi_1)*N_3-eye(20)+(kron((phi_1'*P*B+psi_1'*D)',
    xiu_2*M_2)+kron((phi_1'*P*E+psi_1'*F-gamma^2*W_1')',xiw_2*M_2))*N_3;
Bu_2=B_1*kron(eye(n),U_1*M_1);
Ew_2=E_1*kron(eye(n),W_1*M_1);
Du_2=D_1*kron(eye(n),U_1*M_1);
Fw_2=F_1*kron(eye(n),W_1*M_1);
Gu_2=B'*P*(A_2+Bu_2+Ew_2)+D'*(C_2+Du_2+Fw_2)+m2v(B_1'*P*phi_1+D_1'*psi_1)';
Gw_2=E'*P*(A_2+Bu_2+Ew_2)+F'*(C_2+Du_2+Fw_2)+m2v(E_1'*P*phi_1+F_1'*psi_1)';
etauw_2=RI*[Gu_2;Gw_2]*N_2;
etau_2=etauw_2(1,:);
etaw_2=etauw_2(2,:);
H_3=-m2v([P*phi_1;psi_1;-gamma^2*W_1]'*[(B*etau_2+E*etaw_2)*M_2+A_2+Bu_2
     +Ew_2;(D*etau_2+F*etaw_2)*M_2+C_2+Du_2+Fw_2;etaw_2*M_2])'*N_3;

% get P_3
P_3=H_3*L_3^(-1);

% step 4
% get U_2 and W_2
U_2=P_3*xiu_2+etau_2;
W_2=P_3*xiw_2+etaw_2;

% step 5
% get L_4, H_4 and P_4
phi_2=B*U_2*M_2+E*W_2*M_2+(A_2+Bu_2+Ew_2);
psi_2=D*U_2*M_2+F*W_2*M_2+(C_2+Du_2+Fw_2);
Gamma_33=kron(kron(phi_1,phi_1),phi_1);
Gamma_43=kron(kron(phi_2,phi_1),phi_1)+kron(kron(phi_1,phi_2),phi_1)
           +kron(kron(phi_1,phi_1),phi_2);
B3_4=kron(kron(Gamma_00,B),Gamma_33)+kron(kron(Gamma_11,B),Gamma_22)
     +kron( kron(Gamma_22,B),Gamma_11)+kron(kron(Gamma_33,B),Gamma_00);
E3_4=kron(kron(Gamma_00,E),Gamma_33)+kron(kron(Gamma_11,E),Gamma_22)
     +kron(kron(Gamma_22,E),Gamma_11)+kron(kron(Gamma_33,E),Gamma_00);
xiu_3=M_4*(RI(1,1)*B3_4+RI(1,2)*E3_4)*N_3;
xiw_3=M_4*(RI(2,1)*B3_4+RI(2,2)*E3_4)*N_3;
L_4=M_4*kron(kron(kron(phi_1,phi_1),phi_1),phi_1)*N_4-M_4*N_4
```

```
        +(kron((phi_1'*P*B+psi_1'*D)',xiu_3*M_3)
        +kron((phi_1'*P*E+psi_1'*F-gamma^2*W_1')',xiw_3*M_3))*N_4;
Gamma_32=kron(phi_2,phi_1)+kron(phi_1,phi_2);
Gamma_10=zeros(1,4);
Gamma_21=phi_2;
B3_3=kron(kron(Gamma_00,B),Gamma_32)+kron(kron(Gamma_10,B),Gamma_22)
        +kron(kron(Gamma_11,B),Gamma_21)+kron(kron(Gamma_21,B),Gamma_11)
        +kron(kron(Gamma_32,B),Gamma_00)+kron(kron(Gamma_22,B),Gamma_10)
        +kron(kron(Gamma_00,B_1),Gamma_22)+kron(kron(Gamma_11,B_1),Gamma_11)
        +kron(kron(Gamma_22,B_1), Gamma_00);
E3_3=kron(kron(Gamma_00,E),Gamma_32)+kron(kron(Gamma_10,E),Gamma_22)
        +kron(kron(Gamma_11,E),Gamma_21)+kron(kron(Gamma_21,E),Gamma_11)
        +kron(kron(Gamma_32,E),Gamma_00)+kron(kron(Gamma_22,E),Gamma_10)
        +kron(kron(Gamma_00,E_1),Gamma_22)+kron(kron(Gamma_11,E_1),Gamma_11)
        +kron(kron(Gamma_22,E_1), Gamma_00);
Bu_3=B_1*kron(eye(n),U_2*M_2)+B_2*kron(eye(n^2),U_1*M_1);
Ew_3=E_1*kron(eye(n),W_2*M_2)+E_2*kron(eye(n^2),W_1*M_1);
Du_3=D_1*kron(eye(n),U_2*M_2)+D_2*kron(eye(n^2),U_1*M_1);
Fw_3=F_1*kron(eye(n),W_2*M_2)+F_2*kron(eye(n^2),W_1*M_1);
Gw_3=E'*P*(A_3+Bu_3+Ew_3)+F'*(C_3+Du_3+Fw_3)+m2v(E_1'*P*phi_2+F_1'*psi_2)'
        +m2v(E_2'*P*phi_1+F_2'*psi_1)'+P_3*M_3*E3_3;
Gu_3=B'*P*(A_3+Bu_3+Ew_3)+D'*(C_3+Du_3+Fw_3)+m2v(B_1'*P*phi_2+D_1'*psi_2)'
        +m2v(B_2'*P*phi_1+D_2'*psi_1)'+P_3*M_3*B3_3;
etauw_3=RI*[Gu_3;Gw_3]*N_3;
etau_3=etauw_3(1,:);
etaw_3=etauw_3(2,:);
H_4=-0.5*m2v(phi_2'*P*phi_2+psi_2'*psi_2-gamma^2*M_2'*W_2'*W_2*M_2)'*N_4
    -P_3*M_3*Gamma_43*N_4-m2v([P*phi_1;psi_1;-gamma^2*W_1]'*[(B*etau_3
    +E*etaw_3)*M_3+A_3+Bu_3+Ew_3;(D*etau_3+F*etaw_3)*M_3+C_3+Du_3+Fw_3;
    etaw_3*M_3])' *N_4;
P_4=H_4*L_4^(-1);

% step 6
% get U_3 and W_3;
U_3=P_4*xiu_3+etau_3;
W_3=P_4*xiw_3+etaw_3;

% get S_4
PM_4=P_4*M_4;
for i=1:4
    TT(i,:)=PM_4((i-1)*64+1:i*64);
    P1_4(i,:)=TT(i,:);
end
```

```
for i=1:16
    TTT(i,:)=PM_4((i-1)*16+1:i*16);
end
for j=1:4
    for k=1:4
        P2_4(j,(k-1)*16+1:k*16)=TTT((k-1)*4+j,:);
    end
end

for i=1:64
    TTTT(i,:)=PM_4((i-1)*4+1:i*4);
end
for j=1:4
    for k=1:16
        P3_4(j,(k-1)*4+1:k*4)=TTTT((k-1)*4+j,:);
    end
end

for i=1:256
    TTTTT(i)=PM_4(i);
end
for j=1:4
    for k=1:64
        P4_4(j,k)=TTTTT((k-1)*4+j);
    end
end
S_4=(P4_4+P1_4+P2_4+P3_4)';

% get the coefficients of controller
Bb=kron(B,eye(16))+kron(kron(eye(4),B),eye(4))+kron(eye(16),B);
Eb=kron(E,eye(16))+kron(kron(eye(4),E),eye(4))+kron(eye(16),E);
Bt=B'*P'*B_2+B'*S_4'*Bb*kron(phi_1,phi_1)+B'*P*B_2;
Et=B'*P'*E_2+B'*S_4'*Eb*kron(phi_1,phi_1)+E'*P*B_2;

% save data which will be used in the rest file
save('data','U_1','U_3','W_3','W_1','P','Et','Bt','N_2','C','D');

% end of the program

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%% Main Program to simulate the closed-loop discrete-time system
```

```
%% File name: main_sn2.m

close all
clear all
% 1st order h infinite controller vs 3rd order h infinite controller in discrete RTAC system

% initial state
x0=[1.5 -1.5 1.5 -1.5]';
i=1;
xlh(:,i)=x0;
ulh(i)=0;
tlh(i)=i;
% disturbance
wlh(i)=1*sin(0.025*i);
h=0.01;
A=h*[ 0 1 0 0;
      -25/24 0 0 0;
      0 0 0 1;
      5/24 0 0 0]+eye(4);
B=h*[0 -5/24 0 25/24]';
E=h*[0 25/24 0 -5/24]';
F=0;
vpsl=0.2;
load('data');


% system simulation with linear h infinite controller
while i < 5000,
      i=i+1;
      xlh(1,i)=h*xlh(2,i-1)+xlh(1,i-1);
      xlh(2,i)=h*((-xlh(1,i-1)+vpsl*xlh(4,i-1)^2*sin(xlh(3,i-1)))/(1-vpsl^2*cos(xlh(3,i-1))^2)
                  +ulh(i-1)*(-vpsl*cos(xlh(3,i-1))/(1-vpsl^2*cos(xlh(3,i-1))^2))
                  +wlh(i-1)*(1/(1-vpsl^2*cos(xlh(3,i-1))^2)))+xlh(2,i-1);
      xlh(3,i)=h*xlh(4,i-1)+xlh(3,i-1);
      xlh(4,i)=h*((vpsl*cos(xlh(3,i-1))*(xlh(1,i-1)-vpsl*xlh(4,i-1)^2*sin(xlh(3,i-1))))/(1
                  -vpsl^2*cos(xlh(3,i-1))^2)+ulh(i-1)*(1/(1-vpsl^2*cos(xlh(3,i-1))^2))
                  +wlh(i-1)*(-vpsl*cos(xlh(3,i-1))/(1-vpsl^2*cos(xlh(3,i-1))^2)))+xlh(4,i-1);
      wlh(i)=1*sin(0.025*i);
      ulh(i)=U_1*xlh(:,i)-(B'*P*B+D'*D)^(-1)*(E'*P*B)*(wlh(i)-W_1*xlh(:,i));
      tlh(i)=i;
end

i=1;
xnh(:,i)=x0;
```

```
unh(i)=0;
tnh(i)=i;
wnh(i)=1*sin(0.025*i);

% system simulation with 3rd order controller
while i < 5000,
    i=i+1;
    xnh(1,i)=h*xnh(2,i-1)+xnh(1,i-1);
    xnh(2,i)=h*((-xnh(1,i-1)+vpsl*xnh(4,i-1)^2*sin(xnh(3,i-1)))/(1
              -vpsl^2*cos(xnh(3,i-1))^2)+unh(i-1)*(-vpsl*cos(xnh(3,i-1))/(1
              -vpsl^2*cos(xnh(3,i-1))^2))...
              +wnh(i-1)*(1/(1-vpsl^2*cos(xnh(3,i-1))^2)))+xnh(2,i-1);
    xnh(3,i)=h*xnh(4,i-1)+xnh(3,i-1);
    xnh(4,i)=h*((vpsl*cos(xnh(3,i-1))*(xnh(1,i-1)
              -vpsl*xnh(4,i-1)^2*sin(xnh(3,i-1))))/(1-vpsl^2*cos(xnh(3,i-1))^2)
              +unh(i-1)*(1/(1-vpsl^2*cos(xnh(3,i-1))^2))...
              +wnh(i-1)*(-vpsl*cos(xnh(3,i-1))/(1-vpsl^2*cos(xnh(3,i-1))^2)))+xnh(4,i-1);
    wnh(i)=1*sin(0.025*i);
    l=0;
    for i1=1:4
        for i2=1:4
                if i1>i2
                    l=l;
                else
                    l=l+1;
                    xnh2(l,i)=xnh(i1,i)*xnh(i2,i);
                end
        end
    end
    l=0;
    for i1=1:4
        for i2=1:4
            for i3=1:4
                if i2>i3 | i1>i2
                    l=l;
                else
                    l=l+1;
                    xnh3(l,i)=xnh(i1,i)*xnh(i2,i)*xnh(i3,i);
                end
            end
        end
    end
    unh3(i)=U_3*xnh3(:,i);
    wnh3(i)=W_3*xnh3(:,i);
```

```
    unh(i)=U_1*xnh(:,i)+unh3(i)-B'*P*E/(B'*P*B+D'*D)*(wnh(i)-W_1*xnh(:,i)
            -wnh3(i))+(E'*P*B*Bt/(B'*P*B+D'*D)^2-Et/(B'*P*B+D'*D))*N_2*xnh2(:,i)
            *(wnh(i)-W_1*xnh(:,i));
    tnh(i)=i;
end

% plot the figures
figure(1)
plot(tnh*0.01,xnh(1,:), 'r-',tlh*0.01,xlh(1,:),'b:');
% legend('Target Angular Velocity','Practical Angular Velocity');
title(['Solid: nonlinear controller, Dotted: linear controller']);
xlabel('Time(s)');
ylabel('x1');


figure(5)
plot(tnh*0.01,unh, 'r-',tlh*0.01,ulh,'b:');
% legend('Target Angular Velocity','Practical Angular Velocity');
title(['Solid: nonlinear control input, Dotted: linear control input']);
xlabel('Time(s)');
ylabel('u');

figure(6)
plot(xnh(1,:), xnh(2,:), 'r-', xlh(1,:), xlh(2,:), 'b:');
title(['Solid: nonlinear controller, Dotted: linear controller']);
xlabel('x1');
ylabel('x2');

figure(7)
plot(xnh(3,:), xnh(4,:), 'r-',xlh(3,:), xlh(4,:), 'b:');
title(['Solid: nonlinear controller, Dotted: linear controller']);
xlabel('x3');
ylabel('x4');


% end of the program

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Some useful Subprograms

```
%% Program to expend a matrix to a vector, function row()
%% File name: m2v.m

function     col = m2v(M)
[mm,qq] = size(M);
rgamma=zeros(mm*qq,1);
for j=1:qq
rgamma((j-1)*mm+1:j*mm,1) = M(1:mm,j);
end;
col=rgamma;
return

% end of program

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%% Program to convert a vector to a matrix,
%% File name: v2m.m

function     mat = v2m(V,m,q)
mat=zeros(m,q);
for j=1:q;
mat(1:m,j) = V((j-1)*m+1:j*m);
end;
return

% end of program

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Vita

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